ARROW'S IMPOSSIBILITY THEOREM

AND

ELECTORAL SYSTEMS

by

Kwok-man Lui

A Thesis Submitted in Partial Fulfillment
of the Requirement for the Programme
of Master of Philosophy,
in the Department of
Government & Public Administration;

The Chinese University of
Hong Kong

Supervisor:
Dr. H. C. Kuan

June, 1989

ABSTRACT OF THESIS

The main theme of this theses is to assess Arrow's Impossibility Theorem and also analyze electoral systems in terms of Arrow's conditions and axioms.

Arrow's five conditions and two axioms are:

1. Unrestricted Domain:
   Every logically possible set of ordering of alternatives are admissible as choices for individuals.
2. Positive Association of Individual and Social Values:
   The social ordering should respond positively to alternatives in individual orderings, or at least not negatively.
3. Independence of Irrelevant Alternatives:
   The social ordering shall not be altered by the presence or absence of an irrelevant alternatives in the set of alternatives.
4. Citizen's Sovereignty:
   A social choice should be permitted with reference to the unanimity of all the individuals in the society.
5. Nondictatorship:
   A social ordering should not be imposed by a dictator of the society, irrespective to the preference of the rest of the society.

Axiom I : Connectivity
   For all x and y, either xRy or yRx.

Axiom II: Transitivity
   For all x, y and z, xRy and yRz imply xRz.

Arrow claims that these conditions and axioms are the necessary conditions for a social choice mechanism to be rational. However, they cannot simultaneously be fulfilled.

This thesis divides into four sections. In section I, Arrow's conditions are presented and a formal proof of the Impossibility Theorem are also provided.

In section II, various conditions and axioms that Arrow imposes on a social welfare function are being critically assessed. It is concluded that Arrow's impossibility result can be avoided by restriction on the Condition of Unrestricted Domain. The cardinal utility scale does not violate the Condition of Independence of Irrelevant Alternatives. Moreover, it is possible to apply the cardinal utility scale meaningfully in some situations. Pareto Principle is a necessary condition for a social choice mechanism, but unanimity rule is not. Infringement of the Nondictatorship Condition is problematic since a choice which is derived from a single individual can hardly be called a social choice at all. The Axiom of transitivity can be derived from the Axiom of connectivity. Moreover, Arrow's requirement of transitivity may not be a necessary condition for collective choice procedures. The violation of the social transitivity can be justified by the populistic democractic theory.

In section III, Arrow's conditions are reformulated and general survey of electoral systems is performed.

In the final section, six major types of electoral systems are analyzed in terms of Arrow's conditions to see which conditions do these systems violate. The analysis of electoral systems are based on the reformulated version of Arrow's conditions. It has found that among the six major types of electoral systems: the plurality systems, two-ballot system, alternative vote system, single non-transferable vote system, list systems and the single transferable vote system. All of them violate the condition of consistency. The single non-transferable vote electoral system even violates the condition of nondictatorship.

CONTENTS

PAGE

SECTION IV: ANALYSIS OF ELECTORAL SYSTEMS

INTRODUCTION

Kenneth J. Arrow's Impossibility Theorem is a theorem on social choice. According to A. F. Mackay's description, a social choice situation consists of some choosers, some choice alternatives, some information about the chooser's preferences over the choice alternatives, and an aggregation device that combines the preferences of individuals into a collective choice.[1]

The theory of social choice is to study aggregation devices to see whether there is a perfect, rational , fair or acceptable design for the aggregation of individuals' preferences into a social choice. However, Arrow's Impossibility Theorem shows that given certain 'reasonable' assumptions, there can be no ideally rational aggregation device.

Arrow proves that it is logically impossible to construct a 'reasonable social choice function'. It means that all electoral systems used in our daily life is not a 'rational aggregation device'. Hence, "there is no method of voting which will remove the paradox of voting..., neither plurality voting nor any scheme of proportional representation, no matter how complicated. Similarly, the market mechanism does not create a rational social choice."[2]

The plausibility of the theorem, to a large extent, depends on what is meant by 'rational'. In Arrow's conception of rationality, a rational social choice procedure (which Arrow termed it as a social welfare function) should be able to aggregate individuals' orderings of alternatives into a social orderings. "By a social welfare function will be meant a process or rule which, for each set of individual orderings $R_1, \ldots, R_n$ for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, R."[3]

The individual preference orderings are the only raw material for inputting into the social welfare function to yield a social choice.

Arrow has said, "it will be assumed that individuals are rational, by which is meant that the ordering relation R_i satisfy Axioms I and II. The problem will be to construct an ordering relation for society as a whole that will also reflect rational choice-making so that R may also be assumed to satisfy Axiom I and II."[4]

Hence, a social choice and an individual choice possesses the same essence of rationality.

The necessary condition for a social choice to be rational is to satisfy both Axiom I and II listed in the following:

Axiom I : Connectivity

For all x and y, either xRy or yRx.

Axiom II: Transitivity

For all x, y and z, xRy and yRz imply xRz.[5]

(Where the variables x,y and z stand for the alternatives . The relation, R, stand for 'is preferred or indifferent to'

In addition to the two axioms, Arrow imposes five conditions on the social welfare function, which he claims to be "apparently reasonable conditions" for the design of any rational aggregation device.[6]

These five conditions are:

1. Unrestricted Domain:

Every logically possible set of ordering of alternatives are admissible as choices for individuals.

2. Positive Association of Individual and Social Values:

   The social ordering should respond positively to alternatives in individual orderings, or at least not negatively.

3. Independence of Irrelevant Alternatives:

   The social ordering shall not be altered by the presence or absence of an irrelevant alternatives in the set of alternatives.

4. Citizen's Sovereignty:

   A social choice should be permitted with reference to the unanimity of all the individuals in the society.

5. Nondictatorship:

   A social ordering should not be imposed by a dictator of the society, irrespective to the preference of the rest of the society.

Arrow argues that the above two axioms and the five conditions are the elementary requirements for an aggregation mechanism to fulfill in order to be a rational device. However, Arrow also shows that the two ordering axioms and the five conditions are inconsistent. This important result is known as the 'Arrow's Impossibility Theorem'. The theorem says that no collective choice mechanism can always satisfy these conditions simultaneously.

The problem of social choice, is to define a 'reasonable' method to aggregate individuals' preference to generate a single preference pattern for the society composed of these individuals. Arrow's finding, however, gives a negative answer to the problem. By implication, there are no social choice mechanisms are not fundamentally flawed.

Electoral systems are the major social choice mechanism used in democratic countries to settle political affairs. If Arrow's theorem holds, all electoral systems must violate at least one of Arrow's conditions. This is the challenge presented by Arrow that there is no such thing as a flawless electoral system for choosing one of three or more candidates.

In democratic countries worldwide, there are five main types of electoral systems currently practised: the plurality systems, majority systems, semi-proportional systems, list proportional representation systems and single transferable vote system. What conditions do these electoral systems violate? Is the infringement of Arrow's conditions justifiable? These questions will be tackled later in this thesis.

In the first place, I shall present the 'innocuous' conditions which Arrow thinks that all reasonable individuals would agree. Moreover, I also present a proof for the Impossibility Theorem in the section I. Arrow's original proof and conditions contained certain flaws which were subsequently revised in the second edition of his Social Choice and Individual Values. However, his revised proof seems incomplete (he only proof the theorem for three alternatives, moreover, some steps for the proof are omitted), therefore, the proof given in chapter 2 is a modification of his formulation.

Arrow's Impossibility Theorem is a logical conclusion, not the result of empirical observation, instead, it is the result of the logical deduction from a set of premises (i.e. the five conditions and two axioms mentioned above). An analysis of the formal proof, which is now commonly accepted as correct, shows that this Theorem does derive from his conditions. Therefore, the assessment of the Theorem will only concentrate on Arrow's conditions. A sound argument should fulfill two requirements. First, it must be formally valid. Second, its premises must be justified. Since the Impossibility Theorem is formally valid, the last hope for us to overcome Arrow's challenge is to 'discover' a condition which is not a necessary requirement for a social choice mechanism. Thus, in section II, Arrow's conditions and axioms will be critically assessed.

In section III, Arrow's conditions are reformulated and general survey of electoral systems is performed. These are preparations for section IV.

In the final section, various types of electoral systems will be analyzed in terms of Arrow's conditions to see which conditions do these systems violate. The findings, will be summarized and interpreted in the conclusion.

SECTION I

ARROW'S CONDITIONS AND HIS IMPOSSIBILITY THEOREM

THE CONDITIONS OF ARROW'S IMPOSSIBILITY THEOREM

PROOF OF ARROW'S IMPOSSIBILITY THEOREM

CHAPTER ONE

THE CONDITIONS OF ARROW'S IMPOSSIBILITY THEOREM

§1.1 A Notation for Preferences and Choice

Since 'social choice' is concerned with human preferences among alternatives, We first postulate a set of choosers, M:(1, 2, 3, ... , M); a set of alternatives, A:(x, y, z, ...).

We also postulate that the choosers are able to make an ordering relation, R on the alternatives. R stands for 'preference or indifference'. It allows a chooser to say whether he prefers x to y or is indifferent between them. The relation R, may be thought of the combination of preference, P and indifference, I. In the area of choice, the relation R, is the analogue of the relation of greater than or equal to in the area of numerical comparison; P, the relation of preference, is the analogue of greater than; and the relation of indifference I, is the analogue of equal to.

Now we postulate two Axioms, which Arrow argues to be the necessary conditions for a choice to be rational. These are the notions of rationality.

"Axiom I : For all x and y, either xRy or yRx."[7]

This is the axiom of connectivity that guarantees comparability between members of every pair of alternatives. "[F]or any pair of alternatives x and y, either x is preferred to y or y to x, or the two are indifferent. That is, we have assumed that any two alternatives are comparable."[8]

The axiom does not exclude the possibility of both xRy and yRx holds since x and y may be indifferent.

"Axiom II: For all x, y and z, xRy and yRz imply xRz."[9]

This is the axiom of transitivity. It states that an individual prefers alternative x to y, and y to z, then that individual should prefer x to z; otherwise, he is irrational. The notion of transitivity is the analogue of a mathematical relation which applies to numbers. In the set of real numbers, if one number is greater than a second number, and the second number is greater than a third number, then the first number is greater than the third. Putting it in symbols, for a, b, c are elements of real number, we have the relation:

If $a>b$, and $b>c$;

then $a>c$.

This is the law of transitvity in inequality.

Arrow claims that this logical properties of inequality still holds in the choice action of a rational individual. Together Axiom I and II, guarantee an individual is able to rank alternatives in a complete order. In fact, arrow calls the axioms as the requirements of collective rationality and may be taken as the definition of rationality in collective choice. "[T]he consequences of the assumption of rationality is that the choice to be made from any set of alternatives can be determined by the choice made between pairs of alternatives."[10]

Furthermore, Arrow imposes five conditions on the construction of a social welfare function - an aggregation device for the conversion of individual preferences to a social choice.

§1.2 Condition I : Unrestricted Domain

"All logically possible orderings of the alternative social states are admissible."[11]

The condition means that there is no restriction on an individual in the ordering of his preferences. One would wish to express whatever preferences he has. If there is any restriction, one may be deprived from his genuine preferences. Thus, it is unreasonable to impose any limitations to the choices of individuals.

In political philosophy, this might have been called freedom of choice. This condition applies not only to individual, but also to the collective choice. For n alternatives, there will be n! (that is, 1x2x3x...xn) ways in which the n elements can be ordered. If there are three alternatives, say x, y, z, then there are 3! (1x2x3) possible ways of ordering:

(1) xRy, yRz, and xRz (by transitivity)
(2) xRz, zRy, and xRy (by transitivity)
(3) yRx, xRz, and yRz (by transitivity)
(4) yRz, zRx, and yRx (by transitivity)
(5) zRx, xRy, and zRy (by transitivity)
(6) zRy, yRx, and zRx (by transitivity)

§1.3 Condition II : Positive Association of Individual and Social Values

"For a given pair of alternatives, x and y, let the individual preferences be given. Suppose that x is then raised in some or all of the individual preferences. Then if x was originally socially preferred to y, it remains socially preferred to y after the change."[12]

This condition is to assure some positive association between individual preference and social choice.

Arrow gives a justification for this condition: "Since we are trying to describe social welfare and not some sort of illfare, we must assume that the social welfare function is such that the social ordering responds positively to alterations in individual values, or at least not negatively."[13]

Supposing there were three alternatives, x, y and z, and the society order, xRy. Now if individuals either raise x in their preference order or do not change its position, we have no reason to lower x from the social ordering. "This is a natural requirement since no individual ranks x lower than he formerly did; if society formerly ranked x above y, we should certainly expected that it still does."[14]

§1.4 Condition III : The Independence of Irrelevant Alternatives

"Let $R_1, \ldots, R_n$ and $R_1', \ldots, R_n'$ be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR_i'y$, then C(S) and C'(S) are the same."[15]

This means that if the set of individual preferences on which a social choice function is based does not change (i.e. $xR_iy=xR_i'y$), then the social choice function should remain the same. (i.e. C(S)=C'(S))

The condition imposes certain constraints on the input that an acceptable aggregation device can respond to.

First, the social welfare function should respond only to preference orderings of individuals. This means that the aggregation device should respond only to information concerning what is preferred, and what is indifferent. It should not respond to the intensity of the preference nor other type of information. We can see that Arrow is an 'ordinalist' in the sense that he does not believe the intensity of preference can be expressed quantitatively by numerical terms in a meaningful way. Only ordinal rankings are acceptable among alternatives.

Second, an aggregation device should respond to a restricted class of alternatives. Only those alternatives involved in social orderings are taken into consideration. Thus, if an alternative does not belong to the restricted class, the presence or absence of such an alternative should not alter the social choice.[16]

Considering two voting methods below, the first method satisfies condition III while the second one violates condition III.

Let there be five voters (1, 2, 3, 4, 5), and four candidates (x, y, z, w); also, let the voters rank the candidates in the following ways:

Method I : The Condorcet Criterion

VOTERS	RANK ORDER OF ALTERNATIVES			
	1st	2nd	3rd	4th
1	x	y	z	w
2	x	y	z	w
3	y	z	x	w
4	w	x	y	z
5	y	w	x	z

[17]

The voting is based on majority rules in a series of contests between each pair of alternatives. A candidates is said to be a winner unless he defeats all other candidates in the pairwise contest. In this case, we can see from the above table, x is the Condorcet winner.

COMPARING x WITH:	x IS PREFERRED BY A MAJORITY COMPOSED OF:
y	1, 2, 4
z	1, 2, 4, 5,
w	1, 2, 3

x is able to defeat all of y, z, w in pairwise contests. It is the Condorcet winner. Deletion of any other alternatives will not change the winner. Condorcet criterion satisfies condition III is obvious since in a pairwise contest, only the two relevant alternatives and their orders , not the irrelevant ones, are taken into consideration.

Method II : The Borda Criterion

VOTERS	CANDIDATES			
	x	y	z	w
1	4	3	2	1
2	4	3	2	1
3	2	4	3	1
4	3	2	1	4
5	2	4	1	3
Totals	15	16	9	10

[18]

Among the four candidates, let the highest rank order have four points, the next have three points, and so on. The voting is based on the summation of points in the rank orders of alternatives in the voters orderings ; and the alternative with the largest sum will be selected as the winner.

We can see that y obtains highest points and hence the winner by the Borda criterion.

Suppose we now delete candidates, say, z and w, and retain x and y in the contest. The candidate in the higher rank now gets two points and the one in lower rank gets one points. The result of the voting thus becomes:

VOTERS	CANDIDATES	
	x	y
1	2	1
2	2	1
3	1	2
4	2	1
5	1	2
Totals	8	7

We can notice that x now becomes the winner. The elimination of z and w alters the winner. The Borda criterion violates condition III.

Arrow insists that the acceptable majority rule should satisfy at least the Condorcet criterion. "Knowing the social choices made in pairwise comparisons in turn determines the entire social ordering and therefore with the social choice function."[19]

Violation of condition III may permit the manipulation of the outcomes by mere introduction or deletion of candidates causing the Condorcet winner to be defeated.

§1.5 Condition IV : Citizens' Sovereignty

"The social welfare function is not to be imposed."[20] "A social welfare function will be said to be imposed if, for some pair of distinct alternatives x and y, xRy for any set of individual orderings $R_1, \ldots, R_n$, where R is the social ordering corresponding to $R_1, \ldots, R_n$."[21]

If a social welfare function is imposed, say xRy, then the society can never choose y over x whatever how many individuals prefer y to x. It means some preferences are forbidden. (Notice that if xRy holds, then yPx does not hold.

If condition IV is violated, a social ordering can not be preferred by the society even all individuals prefer it. "We certainly wish all choices to be possible if unanimously desired by the group."[22] Hence, the reason for holding the condition is, to ensure the will of all individuals in a society is adopted whenever it conflicts with a supposed general will or customary choice.

§1.6 Condition V : Nondictatorship

"The social welfare function is not to be dictatorial."[23] "A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xP_iy implies xPy regardless of the orderings $R_1, \ldots, R_n$ of all individuals other than i, where P is the social preference relation corresponding to $R_1, \ldots, R_n$."[24]

The reason for this condition is obvious. Since we are aiming at construction of an aggregation device, we would like to take all individuals's preferences into consideration. If one individual's preferences can determine the social outcome regardless of the preferences of all others, the choice is not a social choice at all.

§1.7 Arrow's Impossibility Theorem

Arrow asserts that the five conditions mentioned above are "apparently reasonable conditions on the construction of a social welfare function". He says, "taken together they express the doctrines of citizens' sovereignty and rationality in a very general form".[25] Arrow also states that the five conditions together with two ordering axioms cannot be satisfied simultaneously by any summation method. There is an inconsistency among these conditions. And he asserts: "If there are at least three alternatives which the members of the society are free to order in any way, then every social welfare function satisfying conditions 2 and 3 and yielding a social ordering satisfying Axioms I and II must be either imposed or dictatorial."[26] This proposition is known as the Arrow's impossibility theorem. It means that every summation method satisfying conditions I, II, III and axiom of connectivity and transitivity must violate either condition IV (citizens' sovereignty) or condition V (nondictatorship).

In Arrow's revised proof,(notes on the theory of social choice, 1963), condition II and IV are replaced by the Pareto principle.

Pareto Principle : "If xP_iy for all i, then xPy."[27] That is to say, "if every individual prefers x to y, then so does society".[28]

Now we try to prove that the Pareto principle is logically implied from condition II and IV jointly:

From condition IV, a social ordering, say yRx, is not imposed with reference to a set of individual orderings R_1, R_2, ..., R_n. Since not yRx is logically equivalent to xPy. In other words, the social ordering xPy can be chosen by the society.

Supposing in the set of individual orderings, R_1, R_2,..., R_n and the social ordering xPy, there are some individuals do not prefer x to y. Now changing all these individuals' preferences by raising x in their preference order over y and thus achieve unanimity. From condition II, the social ordering should respond positively or at least not negatively to the alterations of individual's preferences. The social ordering therefore must continue to be xPy.

If the set of individuals' orderings, $R_1, R_2, \ldots, R_n$ are unanimously prefer x over y; then, from condition I alone, we obtain Pareto principle that the society should prefer x over y.

Pareto principle is actually a unanimity rule which forbid a social ordering of a pair of alternatives contrary to every one's preferences. The contrary social ordering could come from two ways:

(1) a social outcome nobody wants could come out of the method of summation, but this is forbidden by condition II;

(2) a social outcome is imposed from outside the system, but this is prohibited by condition IV.

Since Pareto principle is equivalent to condition II and IV, Arrow's theorem can be demonstrated by showing the inconsistency among conditions I, III, V and Pareto principle.

CHAPTER TWO

PROOF OF ARROW'S IMPOSSIBILITY THEOREM

In this chapter, I will try to prove Arrow's theorem. The proof will consist of two sections. In the first section, I will consider a choice situation where involves three alternatives and more than two individuals. While in section two, I will apply the result obtained from the first section to prove the theorem still holds when the number of alternatives is greater than three.

In the first section, again I will divide it into two parts. In the first part, I will prove that if an individual is a decisive set for any pair of alternative, then he is a decisive set for all pairs of alternatives and thus is a dictator in violation of condition V. In part two of section I, I will try to prove, in fact, a single individual can be a decisive set for a pair of alternative under condition I, III and Pareto principle. Combining part one and two, we can see that condition I, III and Pareto principle logically imply the negation of condition V. Hence, I show that the five conditions are inconsistent.

§2.1 Section 1: Part One

Before we start the proof, we define the notion of a decisive set in the following:

Decisive Set: A set of individuals, V, is said to be decisive for x against y if the social outcome is xPy when every individual in V prefers x to y for all sets of individual orderings.[29]

A decisive set V, may contain majority, minority of individuals or even one individual.

Arrow further makes a distinction between 'decisive' and 'almost decisive'.

(1) Decisiveness: "$x\bar{D}y$ means that x is socially preferred to y whenever (individual) I prefers x to y, regardless of the orderings of other individuals;"
(2) Almost Decisiveness: "xDy means that x is socially preferred to y if individual I prefers x to y and all other individuals have the opposite preference."[30]

It should be noted that $x\bar{D}y$ implies xDy since 'decisiveness' includes the situation of 'almost decisiveness'. Now, we try to prove the following proposition:

Proposition 1: If a single individual 1 is decisive for x against y, where x is not equal to y, then he is decisive for any pair drawn from the three alternatives (x, y, z).

We construct two sets: V_1, which consists of a single individual 1; and V_3, which consists the rest of all other individuals 2, 3, ..., m; where m is greater than or equal to two. Also, let P_1 be the preference of individual 1, P_j be the social ordering of V_3 and P be the social ordering of the whole society.

Step 1

Suppose the following situations:

(1) Individual 1 is almost decisive for x against y. i.e. xDy (Assumption)

Under condition I, the following individual orderings are admissible:

(2) For V_1: Individual 1 prefer x to y and order y over z i.e. xP_1y, yP_1z and xP_1z (By transitivity)

(3) For V_3: All other individual ranks y over z and chooses y against x. i.e. yP_jz, yP_jx.(It should be noted that the preference relation between alternatives x and z has not been specified.)

From the preference structures above, we select the social outcome by means of pairwise comparisons in accordance with the requirement of condition III.

i. Since individual 1 is almost decisive for x against y, thus the social preference is xPy. ....(1.1)

ii. Since both V_1 and V_3 prefer y to z, by Pareto principle the social preference is yPz......................(1.2)

iii. From (1.1), (1.2); We obtain xPy together with yPz. According to the axiom of transitivity, we get the social preference xPz.................................(1.3)

However, only individual 1 prefers x to z, and the social preference is obtained without any consideration about the preferences regarding alternatives x and z of individuals in V_3. The social preference xPz follows from xP_1z alone regardless of the preferences of the rest of individuals in the society. This means that the single individual 1, is also decisive for x against z provided that he is given to be almost decisive for x over y.

i.e. $x D y$ implies $x\bar{D}z$ ....(1.4)

Step 2

Now, supposing that:

(1) Individual 1 is almost decisive for x against z. i.e. xDz (Assumption)

Under condition I, the following individual orderings are admissible:

(2) For V_1: Individual 1 prefer y to x and order x over z i.e. yP_1x, xP_1z and yP_1z (By transitivity)

(3) For V_3: All other individual ranks z to x and chooses y against x. i.e. zP_jx, yP_jx.(It should be noted that the preference relation between alternatives y and z has not been specified.)

i. From xDz, we have xPz..............................(2.1)

ii. By Pareto principle, yP_1x and yP_jx together, we can obtain yPx......................................(2.2)

iii. From (2.1) and (2.2), by transitivity we get yPz...(2.3)

The social outcome yPz is arrived at only from a single individual's preference, yP_1z.

Therefore, xDz implies $y\bar{D}z$........................(2.4)

This means that if an individual 1 is almost decisive for x against z, then he is also decisive for y against z.

Step 3

Again, considering the following situations:

(1) Individual 1 is almost decisive for y against z. i.e. yDz (Assumption)

(2) For V_1: Individual 1 prefer y to z and order z over x i.e. yP_1z, zP_1x and yP_1x (By transitivity)

(3) For V_3: All other individual ranks z to y and chooses z against x. i.e. zP_jy, zP_jx.(It should be noted that the preference relation between alternatives x and y has not been specified.)

We obtain the results,

i. yDz means yPz..................................(3.1)

ii. From zP_1x and zP_jx, we obtain zPx.................(3.2)

iii. (3.1) and (3.2) jointly yields yPx................(3.3)

Hence, yDz implies $y\overline{D}x$..............................(3.4)

Step 4

Let the following situations be given:

(1) Individual 1 is almost decisive for y against x. i.e. yDx (Assumption)

(2) For V_1: Individual 1 prefer z to y and order y over x i.e. zP_1y, yP_1x and zP_1x (By transitivity)

(3) For V_3: All other individual ranks x to y and chooses z against y. i.e. xP_jy, zP_jy.(It should be noted that the preference relation between alternatives x and z has not been specified.)

i. By yDx, we get yPx................................(4.1)

ii. From zP_1y and zP_jy, we obtain zPy..................(4.2)

iii. From (4.1) and (4.2), zPx is resulted..............(4.3)

Therefore, yDx implies $z\overline{D}x$..........................(4.4)

Step 5

Furthermore, supposing that:

(1) Individual 1 is almost decisive for z against x. i.e. zDx (Assumption)

(2) For V_1: Individual 1 prefer z to x and order x over y i.e. zP_1x, xP_1y and zP_1y (By transitivity)

(3) For V_3: All other individual ranks x to z and chooses x against y. i.e. xP_jz, xP_jy.(It should be noted that the preference relation between alternatives y and z has not been specified.)

i. By zDx, we have zPx...............................(5.1)

ii. From xP_1y and xP_jy, it means xPy..................(5.2)

iii. From (5.1) and (5.2), zPy is resulted.............(5.3)

Thus, zDx implies $z\bar{D}y$..........................(5.4)

Step 6

Assuming that:

(1) Individual 1 is almost decisive for z against y. i.e. zDy (Assumption)

(2) For V_1: Individual 1 prefer x to z and order z over y i.e. xP_1z, zP_1y and xP_1y (By transitivity)

(3) For V_3: All other individual ranks y to z and chooses x against z. i.e. yP_jz, xP_jz.(It should be noted that the preference relation between alternatives x and y has not been specified.)

i. zDy means zPy...................................(6.1)

ii. From xP_1z and xP_jz, yields xPz....................(6.2)

iii. From (6.1) and (6.2), xPy is obtained.............(6.3)

Thus, zDy implies $x\bar{D}y$...........................(6.4)

Step 7

Summing up the results obtained through step 1 to step 6, we have:

$xDy \longrightarrow x\bar{D}z$..(1.4)

$xDz \rightarrow y\bar{D}z$ ....(2.4)

$yDz \rightarrow y\bar{D}x$ ....(3.4)

$yDx \rightarrow z\bar{D}x$ ....(4.4)

$zDx \rightarrow z\bar{D}y$ ....(5.4)

$zDy \rightarrow x\bar{D}y$ ....(6.4)

Recalling the fact that $x\bar{D}y$ implies xDy, for all variables x and y, we obtain the following result by substituting other variables for x,y:

$x\bar{D}z \rightarrow xDz$ ....(7.1)

$y\bar{D}z \rightarrow yDz$ ....(7.2)

$y\bar{D}x \rightarrow yDx$ ....(7.3)

$z\bar{D}x \rightarrow zDx$ ....(7.4)

$z\bar{D}y \rightarrow zDy$ ....(7.5)

$x\bar{D}y \rightarrow xDy$ ....(7.6)

Step 8

From (1.4) and (7.1), we have:

$xDy \rightarrow xDz$ ....(8.1)

From (8.1) and (2.4), we obtain:

$xDy \rightarrow y\bar{D}z$ ....(8.2)

From (8.2) and (7.2), we have,

$xDy \rightarrow yDz$ ....(8.3)

From (8.3) and (3.4), we get:

$xDy \rightarrow y\bar{D}x$ ....(8.4)

From (8.4) and (7.3), we deduce:

$xDy \rightarrow yDx$ ....(8.5)

From (8.5) and (4.4), we obtain:

$xDy \rightarrow z\bar{D}x$ ....(8.6)

From (8.6) and (7.4), we have:

$xDy \rightarrow zDx$ ....(8.7)

From (8.7) and (5.4), we get:

$xDy \rightarrow z\bar{D}y$ ....(8.8)

From (8.8) and (7.4), we find:

$xDy \rightarrow zDy$ ....(8.9)

From (8.9) and (6.4), we obtain:

$xDy \rightarrow x\bar{D}y$ ....(8.10)

Putting all the results in a flow chart, we get:

$xDy \rightarrow x\bar{D}z \rightarrow xDz \rightarrow y\bar{D}z \rightarrow yDz \rightarrow y\bar{D}x \rightarrow yDx \rightarrow z\bar{D}x \rightarrow zDx \rightarrow z\bar{D}y \rightarrow$
$zDy \rightarrow z\bar{D}x \rightarrow zDx \rightarrow z\bar{D}y \rightarrow zDy \rightarrow x\bar{D}y \rightarrow xDy.$

We should notice that xDy implies $x\bar{D}y$ from the chart, and recalling the fact that $x\bar{D}y$ implies xDy. This is to say, xDy is logically equivalent to $x\bar{D}y$. Whenever an individual 1 is almost decisive for x against y, he is also decisive for x against y and vice versa.

The above results also show that, given an individual is almost decisive for x against y, he is decisive for the social orderings: xPy, xPz, yPz, yPx, zPx and zPy. For a set of alternatives containing only three elements: (x, y, z), there are all together six ways of pairwise preference relation among the alternatives. They are exactly the same social orderings which is decisive by the individual 1. Hence, we can conclude that if an individual is decisive for any pair of alternative, then he is decisive in all pairs for a set of three alternatives. The proof of part I is thus completed.

§2.2 Section 1: Part Two

Now we proceed to prove the following proposition, in fact, is true:

Proposition 2: Under condition I, III and Pareto principle, there exists a single individual 1, is decisive for x against y, where x is not equal to y, for a set of alternatives containing only three elements: (x, y, z).

By the Pareto principle, there is at least one decisive set for any pair of alternatives drawn from the triplet (x, y, z), namely, the set composed of all individuals.

Among all sets of individuals which are decisive for some pairs of alternatives, we define a set which is a minimally decisive set in the sense that subtration of a single member would make it indecisive for any pair.

First of all, we construct the following sets:

M : The set of all individuals.

V : A minimally decisive set.

V_1: A set contains a single individual of V.

V_2: A set contains the remaining individuals of V.

V_3: A set contains the individuals not in V.

The set of choosers is divided into two groups,the one in minimally decisive set and that not in the minimally decisive set. Moreover, the set of minimally decisive set is also divided into two parts: a single individual and the rest. We use mathematical symbols to represent the relations among the four sets as:

$V_1 \cup V_2 = V$ (read, the union of V_1 and V_2 equals V)

$V \cup V_3 = M$.

There are at least two individuals in M, since V_2 and V_3 cannot both be empty. Otherwise, it is obviously violation of condition V (Nondictatorship).

Now we start the proof of the proposition 2.

By condition I, the following individual orderings are admissible:

1. Every individual in V_1 prefers x against z and z against y.
(i.e. $\forall_i$: $i \epsilon V_1$ xP_iz and zP_iy.)
2. Every individual in V_2 prefers y against x and x against z.
(i.e. $\forall_j$: $j \epsilon V_2$ yP_jx and xP_jz.)
3. Every individual in V_3 prefers z against y and y against x.
(i.e. $\forall_k$: $k \epsilon V_3$ zP_ky and yP_kx.)

Let P_1, P_2, P_3 be the social preferences of the three sets, V_1, V_2 and V_3 respectively.

The orderings of alternatives for the three sets are given below:

For V_1 : xP_1z, zP_1y, and xP_1y (by transitivity)
For V_2 : yP_2x, xP_2z, and yP_2z (by transitivity)
For V_3 : zP_3y, yP_3x, and zP_3x (by transitivity)

This can be recognized as a socially paradoxical orderings when each set containing only one member.

Step 1

As required by condition III, we make a pairwise comparisons among the orderings. Since both V_1 and V_2 prefer x to z and V_1, V_2 together form a minimally decisive set, we thus arrive at the social outcome to be xPz.

The society prefers x to z. i.e. xPz..............(1)

Step 2

Assuming that the social outcome is yPz.

Actually, only V_2 prefers y to z. If the ordering of the society hence becomes yPz, V_2 must be a minimally decisive set. However, this result contradicts to the given that V_2 is not a minimally decisive set. Therefore, our assumption that yPz is the social outcome is false. This means that the social outcome is: not yPz; which is zRy.

The social outcome is zRy (i.e. zPy or zIy)........(2)
(There are three possible relations among y and z. they are: yPz [y is preferred to z], yIz [y is indifferent to z] and zPy [z is preferred to y]. To say 'not yPz' is equivalent to affirm the later two possibilities. The notion 'zPy' means 'either zIy or zPy')

From (1) and (2) and by transitivity, we obtain the social outcome to be xPy. However, only a single individual prefers x against y and the social outcome is xPy. This means that the single individual is decisive for x against y. Hence we have shown the proposition 2 is true. The proof of section 1 of Arrow's Impossibility Theorem is completed.

§2.3 Section 2

We now proceed to consider a set of alternatives containing more than or equal to three elements. The size of the set is unspecified.

We establish the following proposition, and try to prove it by the method of mathematical induction:

P(n): Under condition I, III and Pareto principle, a single individual is decisive for all pairs drawn from the set of alternatives containing n elements. (where n is greater or equal to three.)

Step 1

When n=3.

We have the proposition:

P(3): Under condition I, III and Pareto principle, a single individual is decisive for all pairs drawn from the set of alternatives containing three elements.

This proposition is what we have already proved in Section 1. Hence, P(n) is true for n=3.

Step 2

When n=4

We have the proposition:

P(4): Under condition I, III and Pareto principle, a single individual is decisive for all pairs drawn from the set of alternatives containing four elements.

Consider the set of alternatives to be: (x, y, z, a_4)

There are ${}_4C_3$ possible ways to form triplets from the four alternatives. They are:

$$(x, y, z) \quad , \quad (x, y, a_4),$$
$$(x, z, a_4) \quad , \quad (y, z, a_4).$$

($_4C_3$ reads as 'three combination of four', which is equal to four)

We should notice that each triplet contains at least two elements from the set of alternatives (x, y, z).

From P(3), we know that a single individual at least decisive in one pair of alternatives in every triplet. Also, from the proof in section I, part one, we know that once a single individual is decisive in one pair of alternative, he is decisive for all pairs. We conclude that the single individual is decisive for all pairs drawn from the set of four alternatives. Therefore, P(n) is true when n=4.

<u>Step 3</u>

Now we suppose that P(n) is true for k, where k is a positive integer greater than three. We have the proposition:

P(k): Under condition I, III and Pareto principle, a single individual is decisive for all pairs drawn from the set of alternatives containing k elements.

We try to show that P(k+1) is also true.

Consider the set of alternatives containing k+1 elements to be: $(x, y, z, a_4, a_5, \ldots a_k, a_{k+1})$. There are all together $_{k+1}C_3$ ways of forming triplets from k+1 alternatives of the set. And each triplet contains at least two elements from the set containing k elements: $(x, y, z, a_4, a_5, \ldots a_k)$
From our assumption, P(k) is true. Therefore, a single individual is decisive at least in one pair of alternatives in each triplet formed from the set of k+1 alternatives. Again, from the prove in

part one of Section I: if a single individual is decisive in one pair, he is decisive in all pairs in a triplet. We conclude that a single individual is decisive in all pairs from the set of alternatives containing k+1 elements. i.e. P(k+1) is true.

From the principle of mathematical induction, P(n) is true for all positive integers n, where n is greater or equal to three. We thus prove the proposition P(n).

The prove of Arrow's Impossibility Theorem is completed.

SECTION II

THE ASSESSMENT OF ARROW'S IMPOSSIBILITY THEOREM

THE PROBABILITY OF ARROW'S PARADOX AND UNRESTRICTED DOMAIN

INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND CARDINAL UTILITY SCALE

PARETO PRINCIPLE

NONDICTATORSHIP

THE NOTION OF RATIONALITY

CHAPTER THREE

THE PROBABILITY OF ARROW'S PARADOX AND UNRESTRICTED DOMAIN

Arrow's impressive proof assures us that no social choice mechanism can satisfy all together the five 'reasonable' conditions and the two Axioms. In view of Arrow's impossibility theorem, one may ask: How likely does the paradox may occur? Before I tackle this question, it is advisable to clarify the notion of probability first.

There are two approaches to the concept of probability. One is the logical interpretation of probability, the other is the relative-frequency interpretation. Under the logical interpretation, probability is defined in numerical term as a ratio of the number of times a favorable event could occur to the total number of related events. For example, a perfectly fair die, at any given roll, the occurrence of each faces are equally likely, therefore, the probability of throwing a 6 when a die is cast is 1/6. The logical notion of probability is exactly a statement about the number of times a particular event could occur given a set of related events. This notion bases on the concept of "equiprobability". In the above case, how can we assign to each face of the die an equal possibility to occur? Traditional philosophers and mathematicians normally use the Principle of Indifference as an explanation. This principle says: when we have no reason in believing that any event has higher chance of occurrence than the others, under this situation, we say the related events has equal probability. However, the weakness of the logical interpretation is that it has nothing to say about the reality. In our case, if the die is not fair, that is to say, one face of the die has higher chance of occurrence than the others, the foundation of the logical notion of probability will be collapsed.

The relative-frequency interpretation suggests that probability is defined as the limit of the relative-frequency of a particular event in an infinite sequence of related events. Therefore, a probability statement is a statement about a future event based on past events. For our convenience, we again use the example of casting a die for discussion. When we say the probability of throwing a 6 is 1/6, it is equivalent to say that in an infinite sequence of events of throwing a die, the relative frequency of throwing a 6 is converging to 1/6.[1]

The study of the probability of the voter's paradox is based on the logical notion of probability. It is because empirical studies is unreliable and difficult. There are several reasons for this. In the first place, in actual voting situations, voters are normally asked to indicate their first choice only. Without the individual preference orders, it is impossible to determine whether a paradox exists. Moreover, the manipulation of voting procedures may eliminate the occurrence of a paradox. Finally, a paradox is prohibited by some restrictions imposed on individuals' preference among alternatives.

For the above practical difficulties and distortion, the empirical study of the occurrence of the paradox is unlikely to produce accurate results.[2] Therefore, the approach to the assessment of the significance of Arrow's impossibility theorem is mainly to estimate the logical probability of the voter's paradox. This means that the total number of possible outcomes in a voting situation are compared to the number of possible paradox outcomes. It is assumed that the occurrence of every possible outcomes are equally likely. We first see the case where only three alternatives involved.

§3.1 Probabilities of Arrow's Paradox in Three Alternatives

Now supposing there are three alternatives and three individuals. Each individuals is free to order the three alternatives.

In general, the number of logically possible orderings of a given number of alternatives can be summarized in simple mathematical terms. The number of logically possible preference orderings is denoted by n!, where $n!=n(n-1)(n-2)...(1)$, and n is the number of alternatives. In our case, there are three alternatives, therefore, there are 3! (i.e. 3x2x1=6) ways of possible orderings. When there are n! orderings and m choosers, there are $(n!)^m$ possible ways of arranging individual preference orders. The formula for computing the number of possible preference profiles thus is $(n!)^m$, which means, n! to the m^{th} power. In our case, with three individuals and three alternatives, (i.e. n=3 and m=3.), there are $(3!)^3=6^3=216$ possible preference profiles.

The problem remains is to determine how many of these 216 preference profiles produce a paradox. It has been calculated, out of 216 preference profiles, six involve unanimity, 90 involve a majority selecting the same order of alternatives. On the remaining 120 cases, where all three choosers select different preference profiles, there are 12 cases that involve selection of two basic preference profiles which produce the paradox:

		Classification I	Classification II
Individual:	1.	${}_xP_y$, ${}_yP_z$; (xyz)	${}_xP_z$, ${}_zP_y$; (xzy)
	2.	${}_yP_z$, ${}_zP_x$; (yzx)	${}_yP_x$, ${}_xP_z$; (yxz)
	3.	${}_zP_x$, ${}_xP_y$; (zxy)	${}_zP_y$, ${}_yP_x$; (zyx)

By permutation of three choosers and the two basic preference paradox producing profiles, there are 12 cases involve paradox. They are:

Voters: Orderings

Voters	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
1	xyz	yzx	zxy	zxy	zyx	xyz	xzy	xzy	yxz	yxz	zyx	zyx
2	yzx	xyz	yzx	xyz	yzx	zxy	yxz	zyx	xzy	zyx	xzy	yxz
3	zxy	zxy	xyz	yzx	xyz	yzx	zyx	yxz	zyx	xzy	yxz	xzy

Therefore, there are only 12 paradoxes out of 216 equally possible preference profiles. The logical notion of probability for a paradox is 12/216 = 0.0556. The paradox will occur less than 5.6%. This is a rather low percentage.

However, in most voting situations, the number of choosers are enormous. The problem now we are concerning is: How does the percentage for a paradox change as the number of individuals increases? Since the number of possible preference increases exponentially at a rapid rate, (recalling the fact that number of possible preferences $=(n!)^m$) for instance, with three individuals and four alternatives, (i.e. n=4 and m=3.) the possible preferences become $(4!)^3=(24)^3=13824$, in practice, only a random sample of preference profiles is selected as representative of the whole. The proportion of paradox-producing preference profiles in the sample, is generalized an serviced as an estimation of the proportion in the whole.

Niemi and Weisberg (1968) have obtained a pattern of probability of paradox-producing profiles for three alternatives.[3]

Probability of No Majority Winner for Three Alternatives [4]

Number of choosers	Probability	Number of choosers	Probability
1	0.0000	17	0.0827
3	0.0556	19	0.0832
5	0.0694	21	0.0836
7	0.0750	23	0.0840
9	0.0780	25	0.0843
11	0.0798	...	...
13	0.0811	infinite	0.0877
15	0.0820		

From the above figures, it is interesting to note that the paradox probability is not affected so much by the number of individuals. When the number of choosers is greater than nine, the probability of the paradox is almost the same as when the number of choosers approaches infinity. In fact, as the number of alternatives remains at three, and as the number of choosers increases, the paradox probability rises from 0.056 to a limit of about 0.088. That is to say, no matter of how many choosers are under consideration, the paradox probability never rises to 10 percent for infinite number of choosers when there are only three alternatives. This is relatively small and might not cause much trouble in our daily life. However, one may argue that not all preference profiles are equally likely to happen.

In a real world situation, individual preferences are determined by certain specific social, economic, political and cultural norms. These norms create certain degree of homogeneity in the society. Therefore, the results from Niemi and Weisberg may be unrealistic.

A. k. Sen (1970) is also aware that the assumption of equiprobability is unrealistic, he said, "Niemi and Weisberg

(1968) have obtained general expressions for the probability of there being no majority winner. The results are, however, difficult to apply to all individuals without difference, but depending on the nature of the social alternatives and variations of such things as tastes, class background, etc., of different individuals, the individuals' probability distributions may really differ substantially."[5]

It would be more realistic to assume that some alternatives have better chance of being preferred than the others.

§3.2 Partial Cultures

Now supposing that not all preference orders has equal chance of being preferred. That is to say, some alternatives were apriori more likely to be chosen as preference than others. Then, what would be the paradox probability?

A situation in which some alternatives are apriori more likely to occur than others has been called as partial culture. Before expressing this concept in symbol, I first introduce the notion of probability vector. This notion is a listing of the probabilities for each individual preference orderings. In the case of three alternatives, there are six possible preference profiles. If all these preference profiles are equiprobable, then each has a probability of 1/6 to be chosen as first preference. The probability vector thus is : (1/6, 1/6, 1/6, 1/6, 1/6, 1/6); where each of the factions stands for one preference order.

The probability vector can be used to express any situation of partial culture. For example, in a situation where the preference orders xzy and zxy are prohibited, the probability vector for that partial culture will be :

(xyz, xzy, yzx, yxz, zxy, zyx) =
(1/4, 0, 1/4, 1/4, 0, 1/4)

The above probability vector has been recognized as the single-peakedness preference profiles, which we shall discuss later.

Since the probability of the voter's paradox in partial cultures depends not only on the number of individuals and alternatives, but also on the structure of the probability vector. Therefore, there is no generalized results of the paradox probability. However, in the perspective of uneven distribution of probabilities among alternatives, some partial cultures may result a higher paradox probability than that of the impartial one with the same number of individuals and alternatives. To illustrate this point, let us consider the case of three individuals and three alternatives in the following partial cultures:

Case 1:

The probability vector: (xyz, xzy, yzx, yxz, zxy, zyx) =
(0, 1/5, 1/5, 1/5, 1/5, 1/5)

In this partial culture, one preference order, xyz is prohibited, the remaining five preference orders are assumed to be equiprobable for the convenience of our discussion. With five preference orders and three choosers, there are 5^3=125 possible preference profiles. Recalling the fact that there are only two basic paradox-producing profiles: Classification I(xyz, yzx, zxy), and Classification II(xzy, yxz, zyx). Elimination of the preference order xyz, means that the six paradoxes associated with Classification I becomes impossible. The remaining six paradoxes resulted are from Classification II only. Therefore, the paradox probability for this partial culture is 6/125 = 0.048

Case 2:

The probability vector: (xyz, xzy, yzx, yxz, zxy, zyx) =
(0, 1/4, 0, 1/4, 1/4, 1/4)

Under this partial culture, there are 4^3 = 64 possible preference orders and again 6 paradoxes are resulted. This is because the two preference orders eliminated are from the same basic paradox-producing profiles. We thus obtain the paradox probability = 6/64 = 0.09

Case 3:

The probability vector: (xyz, xzy, yzx, yxz, zxy, zyx) =
(0, 0, 1/4, 1/4, 1/4, 1/4)

In this partial culture, the elimination of xyz and xzy are from Classification I and Classification II respectively. In other words, all paradox preference orders generated from either of the paradox-producing preference profiles are eliminated. In this case, the paradox probability becomes zero. (It can be seen that by a suitable restriction on the admission of preference orders, paradoxes are eliminated)

Case 4:

The probability vector: (xyz, xzy, yzx, yxz, zxy, zyx) =
(0, 1/3, 0, 1/3, 0 1/3)

In this partial culture, there are all together $3^3 = 27$ possible preference orders. Since the preference orders xyz, yzx, zxy belong to the same basic paradox-producing profile. Three paradoxes from Classification II still persist. Hence, the paradox probability = 6/27 = 0.22

From the above cases, we can notice that once a preference order of a paradox-producing preference profile is eliminated, all of the six paradoxes associated with that preference profile are vanished. The further elimination of any other preference order from that profile does not reduce the number of paradox. Moreover, we can see that the paradox probability increases as more preference orders are eliminated. The paradox probability increases from 5/125 (=0.048) to 6/64(=0.09), and to 6/27(=0.22). The probability of the paradox can be as high as 22 percent.

Comparing it with the result 0.056 obtained under the equiprobability assumption, it is concluded that in certain partial culture, the paradox probability could be nearly four times of that in an impartial culture. Under a partial culture, the Arrow's paradox could be significant even when there are only three alternatives and three choosers.

§3.3

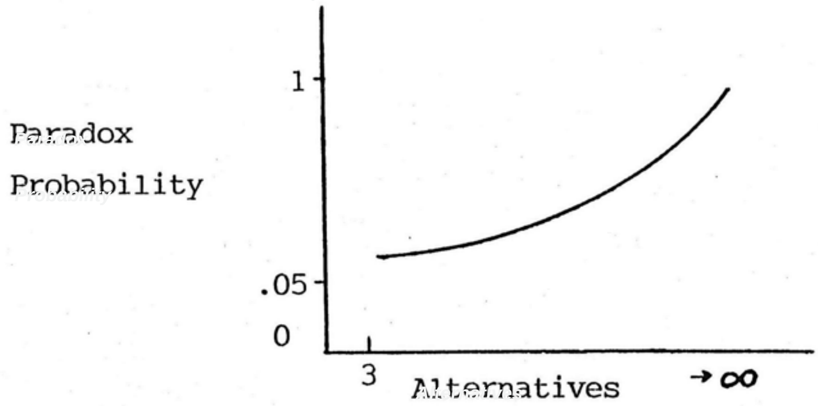
Probability of Arrow's Paradox for More Than Three Alternatives

From the above discussion, we only focus on the paradox probability for three alternatives. However, what would be the paradox probability when alternatives are more than three? Niemi and Weisberg (1968) has worked out the probabilities of no majority for different number of alternatives when the number of voters approaches to infinity.

Limiting Values of Probabilities of No Majority Winner [6]

Number of alternatives	Probability	Number of alternatives	Probability
1	0.0000	20	0.6811
2	0.0000	25	0.7297
3	0.0877	30	0.7648
4	0.1755	35	0.7914
5	0.2513	40	0.8123
10	0.4887	45	0.8292
15	0.6087		

We can notice that, as the number of alternatives increases, the paradox probability approaches one or certainty. If we put the paradox probability as a function of the number of alternatives, we can express their relations in the graph below:



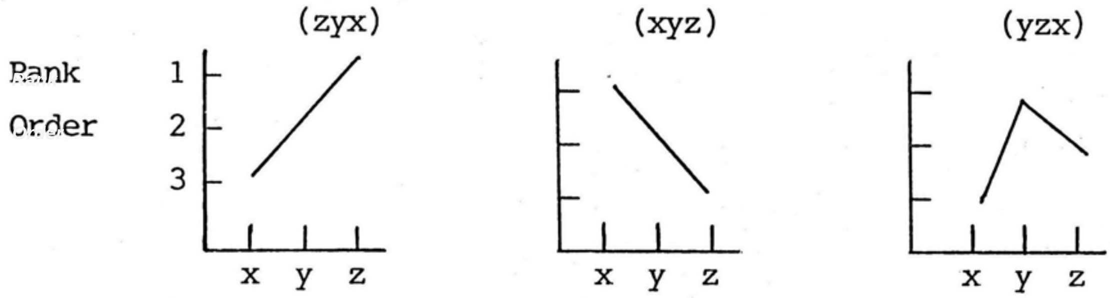
In reality, it is impossible to have infinite number of alternatives. However, under practical consideration, given that the number of alternatives admissible is not more than ten, the paradox probability would be as high as 50%, which can hardly be regarded as insignificant. If in a partial culture, as we have discussed earlier, the occurrence of the paradox might be more severe. In view of this depressing fact, it is advisable to think some ways out of Arrow's paradox.

§3.4 Unrestricted Domain

From the previous discussion we notice that the significance of the probability is mainly due to the increase of the number of alternatives and a certain partial culture. However, we also notice that in a particular partial culture, paradoxes are eliminated completely. This result may give us some idea that certain restriction on the admission of preference orders can reduce the paradox probability to zero. To avoid Arrow's paradox, we should determine what conditions are sufficient for a transitive social outcome. The most well-known and useful condition of sufficiency is proposed by Duncan Black. (1958) This method is known as the single-peakedness method.

§3.5 Single-Peakness Method

This condition is first suggested by Duncan Black. The term 'single-peakedness' is derived from its geometric interpretation. A preference order might be presented in two-coordinate system with the alternatives arranged on a horizontal axis and the rank in the individual chooser's preference order on the vertical axis. For example, the three possible preference orders zyx, xyz, and yzx can be represented as below:



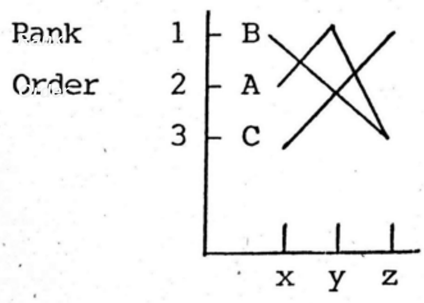
The above curves are all single-peaked. A curve is said to be single-peakedness if

(1): it always slopes up or always down; or

(2): it slopes up a point and then down.

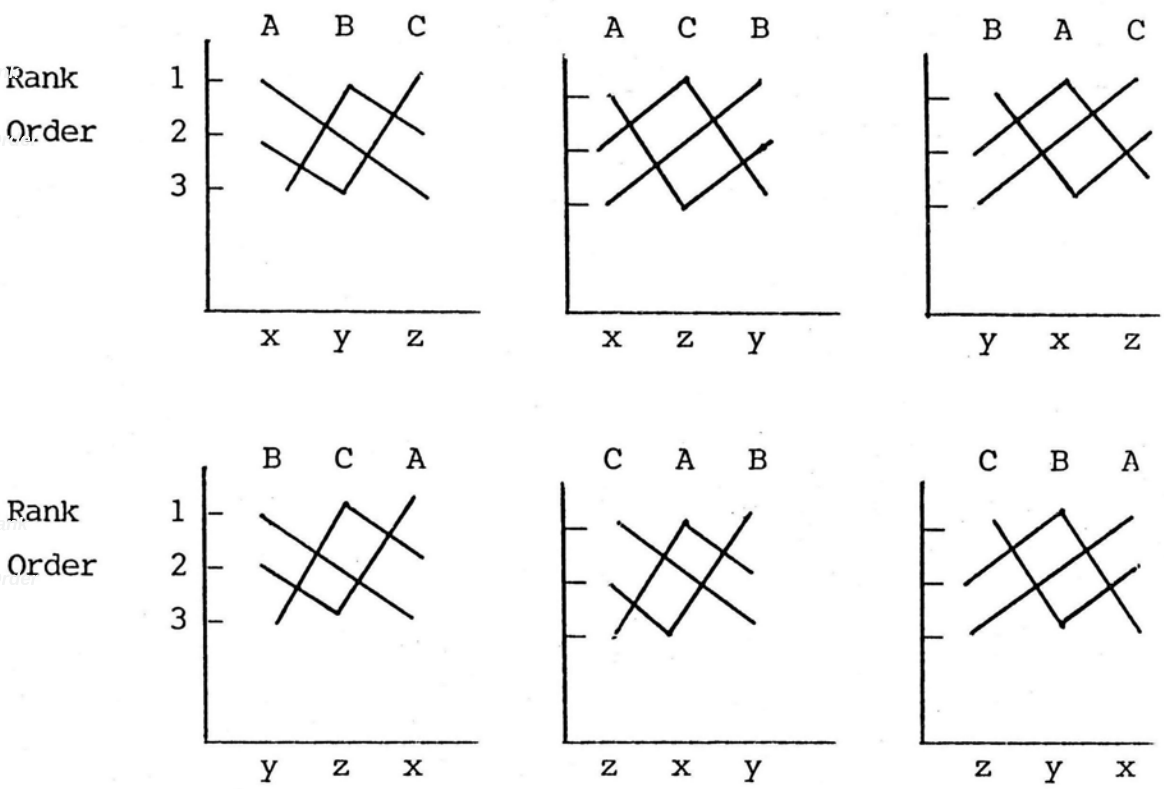
The condition of single-peakedness says that if a set of preference orders can be represented geometrically as a set of single-peaked curves, then there exists some alternatives are preferred by a majority and the social outcomes are transitive.

Let us consider the case of three choosers for three alternatives to illustrate Black's method. Suppose the preference orders for the three individuals A, B, C are xyz, yxz, and zxy respectively. They are graphically represented as:



The possible arrangements of all preference orders as a set of single-peaked curves is the sufficient condition for a transitive social outcome because it eliminates some possible preference orders to be the choice of the choosers.

In the above example, under the requirement of single-peakedness, individual B is not allowed to choose the preference order yzx. If individual B were allowed to choose it, a paradox will be obtained. That is, A:xyz, B:<u>yzx</u>, C:zxy. This preference profiles has six possible arrangement graphically:



They are all nonsingle-peaked set of orderings. We can see that a paradox-producing preference profiles cannot be represented as a set of single-peaked curves under any possible graphical representation. Therefore, single-peakedness method is a limitation on the admission of preference orders, it eliminates all paradox-producing preference profiles. Single-peakedness method states that some alternatives cannot be the last choice of any individual. To see why the restriction of single-peakedness rules out the possibility of the voter's paradox, we first outline the two basic paradox producing profiles: Classification I(xyz, yzx, zxy), and Classification II(xzy, yxz, zyx). Note that each alternatives is in a different rank for each individuals in the two paradox-producing preference profiles. Also notice that each alternatives is ranked as the last choice once at each paradox-producing preference profiles. Elimination of any alternative as the last choice, means the elimination of certain preference orders from both of the paradox-producing preference profiles. Therefore, paradoxes are eliminated.

In our example, if either x or z were eliminated as a possible last choice, one preference order from each of the paradox-producing preference profiles would be eliminated. This makes occurrence of paradox impossible.

§3.6 Value Restrictedness

If one examines the two basic paradox-producing preference profiles displayed previously, one will notice that each alternative appears in every rank order in both the profiles. In view of this structure, Sen (1966) states a fully general sufficient condition for transitivity of social orderings. Sen defines the notion of value-restrictedness as an extension of single-peakedness. A set of individual preference orders is said to be value-restricted if any alternative cannot be the first, second, or third choice for any individual.

Sen proves that majority rule satisfies Condition II - V and transitivity, provided that the set of alternatives is value-restricted.[7] His proof, however, shows that both single-peakedness and value-restrictedness methods are violation of Arrow's condition of unrestricted domain. The paradox thus can be avoided. However, an important question remains unsolved, is the justification for this violation. We cannot infringe the condition simply to avoid paradox.

§3.7 The Infinite Regress Argument

A justification for the infringement of Condition I has been proposed by A. F. Mackay (1980). As he says, "[o]ur new approach will involve showing how both Arrow's result and the paradoxes of majority voting contain instances of an ancient familiar philosophical strategy, viz. the infinite regress argument. Following that we will show how accepting pattern restrictions like single-peakedness can be comparable to positing a first cause the traditional remedy for a regress."[8]

An infinite regress argument is to argue that by assuming some characteristics (eg. transitivity, asymmetry) about a binary relation, an infinite regress will be deduced. Since an infinite regress is unacceptable, something to block the regress must be imposed as to establish a starting point. The argument involves two stages, first, it is to argue that an infinite regress is deduced; second, the positing of a starting point is a necessary remedy.

Mackay first employs Sanford's structure of an infinite regress argument:

> "So long as "R" is interpreted in the same way throughout, any five statements of the following forms are mutually inconsistent:
>
> (1) Existence. $(\exists x)(\exists y)R_{xy}$
> Something exists which has relation R to something.
>
> (2) Asymmetry. $(x)(y)(R_{xy} \supset R_{yx})$
> If x has R to y, then y does not have R to x.
>
> (3) Transitivity. $(x)(y)(z)((R_{xy} \& R_{yz}) \supset R_{xz})$
> If x has R to y which in turn has R to z, then x has R to z.
>
> (4) Existential Heredity. $(x)(y)(R_{xy} \supset (\exists z)R_{zx})$
> If x has R to y, then something has R to x.
>
> (5) Finitude. There are only finitely many things related by relation R." [9]

The above five statements have been proved as mutually inconsistent by David H. Sanford (1975).[10]

Mackay comments: "As Sanford remarks, most traditional uses of the argument assume the first three characteristics (existence, asymmetry, and transitivity) and pit heredity, whose denial implies a first cause, against finitude, whose denial implies a vicious regress....And from existence, heredity, transitivity, and finitude one can deduce nonasymmetry. And so on....we will argue that the paradoxes of majority voting can be viewed in this way. They are applications of infinite regress arguments in which we establish four of these characteristics about the binary relation social preference, and deduce from that the denial of the fifth. [that is the finitude]"[11]

In fact, what Mackay trying to do is to apply Sanford's structure of the infinite argument into Arrow's conditions and Axioms. From those Arrow's conditions, a preference cycle is resulted, which violates the Transitivity. If transitivity is satisfied, there will be infinite regress of preference patterns. Since infinite regress of preference patterns are undesirable, the condition of heredity should be given up. This means, there exists at least one x has R to y, and nothing has R to x. If R_{xy} is the relation that x is majority preferred to y, then violation of the condition of heredity is equivalent to say: there exists one x has majority preferred to y and x is not ranked the last by majority. Thus, the infringement of Arrow's Condition I is analogues to the imposition of a starting point to block an infinite regress. Mackay argues that, "[i]t is hard to feel positing a first cause is an unnatural, or ad hoc way out of an infinite regress."[12] Therefore, "an infringement of unlimited scope as the natural way out. And as before, accepting such a pattern restriction is comparable to positing a first cause, the standard remedy for a regress."[13]

§3.6 Value Restrictedness

If one examines the two basic paradox-producing preference profiles displayed previously, one will notice that each alternative appears in every rank order in both the profiles. In view of this structure, Sen (1966) states a fully general sufficient condition for transitivity of social orderings. Sen defines the notion of value-restrictedness as an extension of single-peakedness. A set of individual preference orders is said to be value-restricted if any alternative cannot be the first, second, or third choice for any individual.

Sen proves that majority rule satisfies Condition II - V and transitivity, provided that the set of alternatives is value-restricted.[7] His proof, however, shows that both single-peakedness and value-restrictedness methods are violation of Arrow's condition of unrestricted domain. The paradox thus can be avoided. However, an important question remains unsolved, is the justification for this violation. We cannot infringe the condition simply to avoid paradox.

Admittedly, Mackay's attempt is fascinating. Nevertheless, I do not think his way of handling Arrow's case is at all promising. First, I would think that Mackay seems to disregard the ethical implication of the infringement of Condition I. Arrow's case is different from other types of infinte regress for its presupposed conditions are value laden. The positing of 'first cause' indeed is ethically problematic. It cannot be justified merely to block an 'infinte regress'. The infringement of Condition I will mean the infringement of certain values which are upheld by Arrow, that is, the freedom of choice. I think, an infringement of a value principle should only be supported by a suitable moral argument. Moreover, even if we assume that the positing of a 'first cause' is permissible, then, immediate questions followed are: Which preference order should be restricted? Who has the authority to decide which social preference must be prohibited? Will this move invite a dictator thus is the violation of Condition V? Finally, if we take the individualistic position seriously, we shall think that the demand for all logically possible preference orders to be permissible is at all natural and desirable. Any infringement of the Condition I is the infringement of values upheld by liberalism. Therefore, the infringement of the unrestricted domain is ethically unjustified. This is why I do not think the alteration of Condition I is promising.

CHAPTER FOUR

INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND CARDINAL UTILITY SCALE

This condition, as suggested by its name, requires that "[t]he choice made from any fixed environment S should be independent of the very existence of alternative outside S".[1]

As we have discussed previously, condition III imposes two constraints on a social mechanism. First, a social choice mechanism should only respond to a restricted class of alternatives. Second, it should respond to preference orderings of individuals only.

In Arrow's original version, he used an example to illustrate the reasonableness of this condition. "In particular, suppose that there are three voters and four candidates, x, y, z, and w. Let the weights for the first , second, third, and fourth choices be 4, 3, 2, and 1, respectively. Suppose that individuals 1 and 2 rank the candidates in the order x, y, z and w, while individual 3 ranks them in the order z, w, x, and y. Under the given electoral system, x is chosen. Then, certainly, if y is deleted from the ranks of the candidates, the system applied to the remaining candidates should yield the same result, especially since, in this case, y is inferior to x according to the tastes of every individual; but, if y is in fact deleted, the indicated electoral system would yield a tie between x and z."[2]

Put Arrow's example graphically, we have the following two cases:

Case 1:

INDIVIDUALS	RANK ORDER OF ALTERNATIVES			
	x	y	z	w
1	4	3	2	1
2	4	3	2	1
3	2	1	4	3
Total:	10	7	8	5

Social preference: ${}_xP_z$, ${}_zP_y$, ${}_yP_w$.

Case 2:

INDIVIDUALS	RANK ORDER OF ALTERNATIVES		
	x	z	w
1	3	2	1
2	3	2	1
3	1	3	2
Total:	7	7	4

Social preference: ${}_xI_z$, ${}_zP_w$, ${}_xP_w$.

In case 2, after elimination of y, the social outcome changes without any corresponding change in the preference orderings of the individuals. Arrow supposes that y is the 'irrelevant alternative' outside the fixed environment S. It is unreasonable to alter the outcome by merely introduction or delection of an 'irrelevant alternative'.

Arrow, not only regards rank-order method unreasonable, he also claims that interpersonal comparison is implausible, therefore, cardinal scaling is meaningless.

In the discussion about the measurability and interpersonal comparability of utility, Arrow said, "[t]he viewpoint will be taken here that interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility." He also added, "[i]f we cannot have measurable utility, in this sense, we cannot have interpersonal comparability of utilities a fortiori."[3]

Therefore, cardinal utility scaling is permissible only when the above challenges can be overcome.

Up to this point, we can see two aspects of argument associated with condition III. One is concerning about the reasonability of alteration of social outcome by the elimination of an alternative from a given environment S. Another aspect is about the plausibility of a cardinal utility scaling. In the following discussion, I shall deal with these two aspects separately. For my convenience, I shall call these two aspects as the independent aspect and the cardinal utility aspect.

§4.1 The Independent Aspect

As we have discussed before, condition III requires that in comparing any pair of alternatives, the social choice mechanism should not respond to information concerning other alternatives not in the choice set. The immediate question is: Which is the 'relevant choice set'? In Arrow's original version, "[i]f S is the set [x, y] consisting of the two alternatives x and y, condition 3 tells us that the choice between x and y is determined solely by the preferences of the members of the community as between x and y.[4] Arrow also claims that, "[k]nowing the social choice made in pair-wise comparison in turn determines the entire social ordering and there with the social choice function C(S) for all possible environments"[5] Thus, a relevant choice set is confined to just two alternatives in each instances. But why should we restrict our choice and comparison only between two alternatives and not involve more than two? Arrow, seems to take it for granted and has no reply. However, on the contrary Fishburn (1973) suggests that a decision mechanism should compare all of the alternatives simultaneously, rather than two at a time. In this way, some intransitive social outcomes from an ordinal preference profile can thus be dissolved. Suppose we use 'cardinal utility' in a preference profile which yields paradox:[6]

INDIVIDUALS	RANK ORDER OF ALTERNATIVES		
	x	y	z
1	10	9	8
2	1	10	7
3	6	5	7
Total:	17	24	22

Social preference: yPz, zPx.

Notice that if we consider the preference order only, a paradox is resulted. However, if we consider cardinal utilities, alternative y will be the social outcome, z the second, and x the last. Cardinal scaling takes all the preferences into consideration and has the advantage to dissolve paradox.

Plott (1967) has pointed out, Arrow's example is erroneous. Plott shows that the two sets of alternatives are not the same in each case as demanded by the formal definition of the condition. In the first case, the given environment S, is (x, y, z, w), where y is included; while in the second case, the environment S is (x, z, w), where y is excluded. Strictly speaking, the condition is not violated in Arrow's example. The change of the choice set is not due to the presence or absence of 'irrelevant alternative' outside the given environment, it is because the change in ranks points assigned to each alternatives. The alteration of the social outcome is a response to a different set of weights to each alternative.

Ray (1973), in his article "Independence of irrelevant Alternatives" adds that "Arrow himself seems to confuse his condition with that of the Radner-Marschak condition of independence of irrelevant alternatives. He illustrates his condition 3 (IIA(A)) by giving an example which violates IIA(R-M)"[7]; Here IIA(R-M) is Radner-Marschak's condition of independence of irrelevant alternatives which is defined as: "if x is an element of choice set of S and belongs to S_1, i.e., $x \in C(S)$ and $x \in S_1 \subset S$ together imply $x \in C(S_1)$."[8] For further disscusion, we first define the following notations: X is the universal set of alternatives assumed to be finite; S is a subset of X; R_i is a social ordering on X held by individual i; R_i^S is the relation R_i over a subset S of X; u is a preference profile for a set of individuals, u=[R_1,...,

R_n]; C(u,S) is the choice set of S with respect to u; r_i is a ranking function that associates an integer with each alternative for a preference ordering R_i; $r_i(x)$ is the number of alternatives strictly preferred by x. Given a profile u=[R_1, ...,R_n], there is a ranking function r given by $r(x)=\sum_i r_i(x)$; r_i^S is a ranking function that associates an integer with each alternative in a subset S of X for a preference ordering R_i, a social choice rule: the Global Borda Rule, is defined as: C(u, S)=[x: $r(x) \geq r(y)$ for all x, y $\in$ S]; $r_i^S(x)$ is the number of alternatives also in a subset of X that are strictly preferred by x. Given a profile u=[R_1,..., R_n], there is a ranking function r given by $r^S(x)= \sum_i r_i^S(x)$; a social choice rule: the Local Borda Rule, is defined as: C(u, S)= [x: $r^S(x) \geq r^S(y)$ for all x, y$\in$S].

We now consider what Arrow's example is trying to illustrate and see why rank-order method does not violate Arrow's condition III by the following cases:

The Global Borda Rule:

Case 3:

For a universal set of alternatives X=[x, y, z, w].
Given a preference profile for a set of individuals: u=[R_1, R_2, R_3], where R_1: xPyPzPw; R_2: xPyPzPw; R_3: zPwPxPy.
(notice that it is exactly the preference profile used in Arrow's example)
We calculate the ranking functions:

alt.	r_1	r_2	r_3	$r=\sum r_i$
x	3	3	1	7
y	2	2	0	4
z	1	1	3	5
w	0	0	2	2

For this profile u we would have, C(u, [xyzw])=[x], C(u, [xzw])=[x].

Case 4:

We will use the same global Borda rule to find a choice set for another preference profile u'

For a universal set of alternatives X=[x, y, z, w].

Given a preference profile for a set of individuals: u'=[R_1', R_2', R_3'], where R_1': xPzPyPw; R_2': yPxPzPw; R_3': zPyPwPx.

We calculate the ranking functions:

alt.	r_1	r_2	r_3	$r=\Sigma r_i$
x	3	2	0	5
y	1	3	2	6
z	2	1	3	6
w	0	0	1	1

For this profile u' we would have, C(u', [xyzw])=[y, z], C(u', [xzw])=[z].

Here we should notice that we are getting different choices from the same environment S: S=[x, z, w] at u and u' although the two profiles are the same on the given environment S.(It can be seen clearly by erasing all alternatives not in the environment [here only alternative y], i.e. $R_i{}^S=R_i{}'^S$.)

Since $x R_i y = x R_i{}' y$ for the given environment [x, z, w] but C(S) is not the same as C'(S). It shows that the global Borda rule violates the condition III.

Now, let us consider the local Borda rule:

The Local Borda Rule

Case 5:

For a universal set of alternatives X=[x, y, z, w].
Given a preference profile for a set of individuals: u=[R_1, R_2, R_3], where R_1: xPyPzPw; R_2: xPyPzPw; R_3: zPwPxPy. (notice that it is exactly the preference profile used in Arrow's example)

i) For the given environmentS=[x, y, z, w]
We calculate the ranking functions:

alt.	r_1^S	r_2^S	r_3^S	$r^S = \Sigma r_i^S$
x	3	3	1	7
y	2	2	0	4
z	1	1	3	5
w	0	0	2	2

C(u, [xyzw])=[x].

ii) For the given environment S=[x,z,w]
We calculate the ranking functions:

alt.	r_1^S	r_2^S	r_3^S	$r^S = \Sigma r_i^S$
x	2	2	0	4
z	1	1	2	4
w	0	0	1	1

C(u, [xzw])=[x, z].

Case 6:

We will use the same local Borda rule to find a choice set for another preference profile u'

For a universal set of alternatives X=[x, y, z, w].
Given a preference profile for a set of individuals: u'=[R_1', R_2', R_3'], where R_1': xPzPyPw; R_2': yPxPzPw; R_3': zPyPwPx.

i) For the given environmentS=[x, y, z, w]
We calculate the ranking functions:

alt.	r_1	r_2	r_3	$r=\Sigma r_i$
x	3	2	0	5
y	1	3	2	6
z	2	1	3	6
w	0	0	1	1

For this profile u' we would have, C(u', [xyzw])=[y, z].

ii) For the given environment S=[x,z,w]
We calculate the ranking functions:

alt.	r_1	r_2	r_3	$r=\Sigma r_i$
x	2	2	0	4
z	1	1	2	4
w	0	0	1	1

For this profile u' we would have C(u', [xzw])=[x, z].

Here for a given environment S=[x, z, w], $R_i^S=R_i'^S$.

Since $xR_iy = xR_i'y$ for the given environment [x, z, w] and C(S) is the same as C'(S). It shows that the local Borda rule satisfies the condition III.

Thus, we can see that the erroneous of Arrow's example is that he first uses the global Borda rule to derive a rank-order of alternatives from u over all X and obtains a choice set for the environment [x, z, w]. After that, he uses the local Borda rule to derive a choice set for the environment [x, z, w]. Arrow is, in fact, demanding the choice obtained from the global Borda rule as the same from that of the local Borda rule for the same environment, but it is true only when the environment is the same as the universal set of alternatives, i.e. S = X.

In fact, the independent of irrelevant alternatives condition in Arrow's mind is that: If a voting rule chooses x as a winner, and some alternative y is then removed from the set of alternatives, the voting rule should still choose x as a winner. i.e. $x \in C(X) \rightarrow x \in C(S)$ for $x \in S \subset X$ and $y \in X$ but $\notin S$. This condition, however, is quite different from the one given by Arrow in the definition.

From the above analysis, the alteration of social outcome is acceptable, it does not violate condition III. However, it does not mean that the cardinal utility scaling is then well justified, we have to go further into a more fundamental problem: The plausibility of interpersonal comparison of cardinal utilities.

§4.2 The Cardinal Utility Aspect

Arrow charges cardinal scaling as meaningless since there is no objective standard for interpersonal welfare comparison. I think, this is the fundamental problem cardinalist has to face.

Cardinal utilities can be thought as units on an interval scale. For example, suppose there are three alternatives with the following utilities for an individual: x=10, y=5, z=3. That individual prefers x to y by 5 utiles, and y to z by 2 utiles. However, if we now add 10 utiles to the original units of the three alternatives respectively, we can obtain the same interval of utiles between alternatives but with different ratio. This shows that the scaling is arbitrary. Thus, we can say that an individual prefers one alternative to another by 5 utiles, but cannot speak meaningfully that one alternative brings twice the utility of another. This is the root of the problem of interpersonal comparisons of utility.

John von Neumann and Oskar Morgenstern (1953), are the first who suggested a cardinal-utility indicator for interpersonal comparison. They employ the notion of 'lottery' in constructing the indicator. In a lottery, we are risking some money gambling for a prize. Normally, we either receive nothing or an amount substantially great. In other words, we pay a little some of money gambling for a relatively large prize with a relatively low chance of winning. The combination of low risk and potential high return counterbalances the relatively low probability of winning. Now, let us see how this notion works.

Suppose that an individual has a lung canncer. He has to choose between a full recovery or one year life-span. His doctor tells him , if he does not have surgery, he will have only one year to live. The surgery, however, is risky, at best he could have a full recovery, but at worst, he may die. Now suppose that the doctor says to that man, that initially the chances of full recovery are 0.8, however, a complication is discovered and the chance of full recovery reduces to 0.6. If at that point the individual changes his mind and decides against surgery. The probability at which that the individual would no longer choose that alternative which is more preferred can be taken as a measure of the intensity of preference. The justification for this method is that an individual has a very strong preference for one alternative over another, he would be willing to take a chance on achieving it even when the chance of success is small. The intensity of the preference is inversely proportional to the chance of achieving it. This is a ratio scale. Moreover, the indicator is confined to the range between zero and one. The valué is not arbitrary and can have a meaningful interpretation. This breakthrough enables interpersonal comparison plausible. However, its application is quite limited, it only can be applied to situations of uncertainty. When the outcomes or the implementation of the choice is certain, the intensity of preference cannot be measured through this method.

Another supplementary method for interpersonal comparison is to assume that there is some commodities , usually money, which has the same value to all individuals, and then we use this commodity to measure other utilities. Obviously, the assumption that money has the same value to all individuals is certainly incorrect. However, if a group of individuals are economically and psychologically homogenous, the use of money as an objective standard for interpersonal comparison may not be unacceptable.

L. Dubins (1977) proposes to use money to bid for alternatives, the amount the individual willing to pay for achieving that alternative will be regarded as an indicator for the intensity of his preference. In this voting method, each voter are asked to bid for each alternative, and the alternative with the highest total bid are chosen to be the winner. Voters are allowed to bid for a positive or negative amount, but his bid has to add up and sum to zero. Collecting the bids made for the winning alternative is followed. We collect the same amount of money a voter bids. 'Collecting' a negative bid means paying the voter that amount. Since the winning alternative will have a positive total bid, we shall always collect more than we pay out.

For example, there are five individuals and three alternatives:

INDIVIDUALS	ALTERNATIVES		
	x	y	z
1	20	10	-30
2	-20	-30	50
3	30	-35	5
4	-10	25	-15
5	-40	15	25
Total:	-20	-15	+35

Social preference: $_zP_x$, $_xP_y$.

From the above preference profiles, individual 1 is equivalent to say that he is indifferent among the three choices: having alternative x win and pay \$20, having y win and pay \$10, and having z win and being paid \$30. In this example, z is the winning alternative. \$50 would be collected from individual 2, \$5 from individual 3, \$25 from individual 5. \$30 of the collected money would give to individual 1 for compensation, and \$15 to compensate individual 4. We can see that individuals prefer alternative z have to pay for it, and individuals dislike it are compensated. The remaining \$35 represents the cost of decision making. However, this method will induce insincere voting. Suppose the preference profiles in the above example represent sincere bids, and consider how individuals might be tempted to bid insincerely. Individuals 2 and 5 get their first choice by paying $50 and $25 respectively. This is considerably a large amount. If they are sure that z will win by a good margin, they can save money by bidding less. Thus, individual 2 could save money by bidding x: -5, y: -15, z: 20. In this way, he can save $30:

INDIVIDUALS	ALTERNATIVES			
	x	y	z	
1	20	10	-30	
2	-5	-15	20	(insincere)
3	30	-35	5	
4	-10	25	-15	
5	-40	15	25	
Total:	-5	0	+5	

Similarly, individual 5 could also save $10 while still obtaining what he wants by bidding x: -25, y: 10, z: 15:

INDIVIDUALS	ALTERNATIVES			
	x	y	z	
1	20	10	-30	
2	-20	-30	50	
3	30	-35	5	
4	-10	25	-15	
5	-27	12	15	(insincere)
Total:	-7	-22	+25	

Now consider what would be the outcome if both individual 2 and 5 bid insincerely

INDIVIDUALS	ALTERNATIVES			
	x	y	z	
1	20	10	-30	
2	-5	-15	20	(insincere)
3	30	-35	5	
4	-10	25	-15	
5	-27	12	15	(insincere)
Total:	8	-3	-5	

Social preference: $_xP_y$, $_yP_z$.

In this situation, both individuals have lost; and the money got from compensation is less than it was worth to them to have x instead of z. Moreover, an optimal choice cannot be made because of insincere bidding.

In fact, if we collect the bids made for the winning alternative, everyone will be tempted to underbid for the alternative he believes will win. If there are some individuals underbid the 'winning alternative', this alternative may loss, and an inferior alternative may be selected. Every individuals may be tempted to "free ride" and let the others pay for the 'winning alternative' which he prefers. This may end up with undesirable outcome.

Up to this point, we can see that even the interpersonal comparison problem can be solved in certain environments, cardinalists, however, still have a tough work at hands. Cardinal utility scaling is vulnerable to strategic manipulation easily. This is what Arrow intends to prevent by imposing condition III to a social choice mechanism. It is not until recently that a scheme for sincere revelation of preferences was developed. This scheme was first discovered by Edward H. Clarke (1971),and more completely developed by Groves and Ledyard (1977). The application of this idea to voting by bid was first made by Tideman and Tullock (1976). This is called the Preference Revealing Process.

§4.2.1 A Preference Revealing Process

The essence of this process is that each individual is offered a chance to change the social outcome that would occur without his vote by paying a special charge equal to the net cost to others that results from including his vote in the decision. A special charge: Clarke tax, is collected. The charge on any one person will not be paid to any other person, a sum of money will be resulted. This represents the cost of the decision making.

As in the example before, each individual will be asked to submit bids for the alternatives, and again, individuals are asked to bid for all alternatives sum to zero. In this time, we shall not collect from each individual the amount he bid for the winning alternative, but instead that amount of his bid which made a difference to the outcome. Consider the preference profiles of the previous example again, where alternative z is the winner:

INDIVIDUALS	ALTERNATIVES (bids in dollars)			Clarke Tax
	x	y	z	
1	20	10	-30	
2	-20	-30	50	
3	30	-35	5	
4	-10	25	-15	
5	-40	15	25	
Total:	-20	-15	+35*	
	Total without Indicated Individuals			
1:(2+3+4+5)	-40	-25	+65*	0
2:(1+3+4+5)	0	+15*	-15	30 (=15-(-15))
3:(1+2+4+5)	-50	+20	+30*	0
4:(1+2+3+5)	-10	-40	+50*	0
5:(1+2+3+4)	+20*	-30	+10	10 (=20-10)
		(*=winner)		40 (Total)

From the above preference profiles, we can see that if without individual 2, the outcome would have been x: 0, y: 15, z: -15. Alternative y would have beaten alternative z by $30. We charge individual 2 a tax of $30. Similarly, for other individual the procedure is the same. For individual 4, without him the outcome would have been x: -10, y: -40, z: 50. The alternative y still would have won. Individual 4 did not affect the outcome at all. Thus, there is no charge to him.

The general procedure is this. For example, with individual 5 alternative z is chosen, but without individual 5 another alternative, x, would have won, beating z by $10. Individual 5's Clarke tax is $10.

Under this scheme, individuals who prefer the winning alternative still have to pay, but they now only pay the amount by which they influenced the decision. The moral justification for this charge might be that each individual is paying exactly the amount by which his participation reduced the total utility of others. Losers are not compensated. If it does, it may induce individuals to place insincere negative bids on the winning alternative. This can destroy the preference revealing incentive of the scheme. The total $40 collected in Clarke taxes represents the cost of decision making.

The virtue of this system is that it motivates sincere revelation of preferences. For instance, consider individual 5 whether it will be benefit for him to overstate the true amount by which he favors, say, alternative z over x (the true amount is $65). This would only be useful if z would not beat x when individual 5 state his true difference of $65, but would beat x only if he exaggerated his difference. But if this were the case, the amount he pays may exceed $65, the amount at which he is unwilling to pay for his preference. (It is because in this case the rest of individuals prefer x to z, only individual 5 prefers z to x and his vote changes the social outcome. Individual 5 is the only one has to pay the compensation for other's loss.) Certainly he will pay more than the original amount, $10, for his favor. Therefore, if individual 5 obtained his preference by overbidding, his Clarke tax would be more than obtaining that preference is worth to him.

Would individual 5 wish to understate his preference for z over x? The only motivation for doing this will be to try to save money. But if z continues to win, individual 5's Clarke tax remains unchanged. The tax he pays is determined independently of the amount he bids since the charge depends only on the total amounts that the other individuals bid. Hence, he could only save money if his underbidding caused alternative z defeated. However, his saving would be less than the amount he values alternative z over other alternatives. Similarly, all individuals will be motivated to state sincerely their preference difference between any two alternatives. Thus, the problem of insincere preference revelation is solved. No individual has to worry about what others will do, and a socially optimal decision using intensities of preferences is arrived. However, we have to make several assumptions before we can reasonably apply this method in real situations. We have to assume that individuals do not wish to hurt others, they vote out of self interests, and that individuals do not form coalitions. Moreover, individuals under considerations are economically and psychologically homogeneous, so that the use of dollars to measure preference intensities is reasonable.

In view of this scheme, we can more clearly see why rank order method does not violate condition III. For instance, if we delete alternative x from the choice setting, what difference would it make? Consider individual 5 first, originally he reports $10 for z over y. If x is dropped from consideration, individual 5 would no longer offer $65 for z against x. He would save much money and become richer, and he might spend more of his wealth to increase his demand for z instead of y, say, from $10 to $30 (e.g. y: -15, z: 15). Similarly, other individuals will also reassign their bids to alternatives. Such wealth redistribution effects could conceivably change the result. x is not an 'irrevelent alternative ', its presence or absence affects the wealth of all individuals.

After going through the above lengthy discussion, I think we can reasonably conclude that cardinal utility scaling does not violate condition III as Arrow claims. It only violates Arrow's intuitive thinking that only ordinal information can be compared meaningfully between individuals. From a theoretical perspective, cardinal information is more precise and useful. In principle, cardinal scale is better than ordinal scale. It should be more desirable if we can use cardinal scale instead of ordinal scale. Cardinal utility scale has been regarded as 'meaningless' since it is associated with several practical difficulties. One problem has been the introduction of a meaningful indicator for measuring the intensity of preference. Another fundamental problem is the feasibility of interpersonal comparison of utility. Lastly, is the problem of enforcing honesty. After all these obstacles have been removed, we can see that the adoption of voting method using intensities of preference in certain situations is acceptable and well justified.

Although in some cases cyclical majority do not exist when cardinal utilities are used, it can not solve the problem of Arrow's paradox completely. Let us consider the following case:

INDIVIDUALS	ALTERNATIVES		
	x	y	z
1	20	10	5
2	5	20	10
3	10	5	20
Total:	+35	+35	+35

Each of the alternatives from the above preference profiles produces a total of 35 utiles. Here the cardinal utility scaling does not dissolve the intransitve outcome. The use of cardinal utility would avoid cyclical majorities in all other situations in which the distribution of utility were not symmetrical. This enables cardinal scaling severely weaken the occurrence of paradox. However, Sen (1970) and Schwartz (1972), have obtained impossibility results similar to Arrow's by using cardinal utility scaling. Therefore, it may concluded that, even cardinal utility is acceptable, it may not overcome the Arrow's paradox completely.

CHAPTER FIVE

PARETO PRINCIPLE

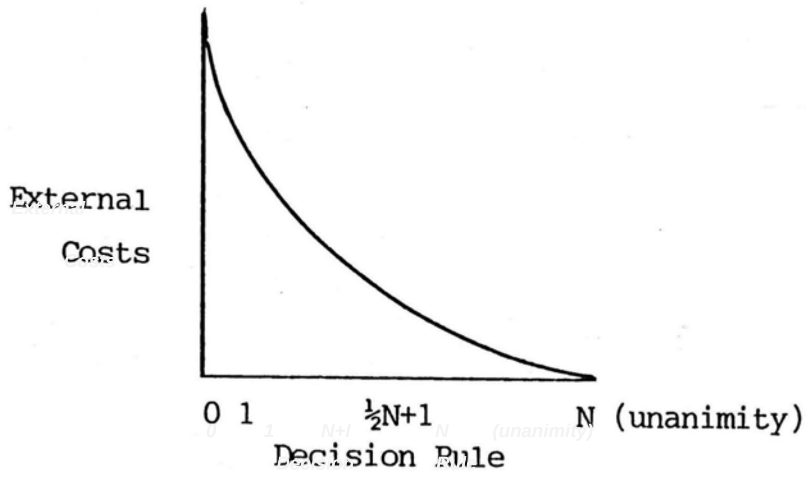
It is a reiteration of the condition II and IV, namely, the condition of positive association of individual and social values and citizens' sovereignty. Pareto Principle requires that if every individual prefers x to y, then the society must prefer x to y. Formally, the Pareto Principle is stated as: "CONDITION P: If xP_iy for all i, then xPy."[1]

The justification for the Pareto Principle is implied from the meaning of 'social choice'. If a choice is unanimously preferred by everyone in the society without exception, it is absurd to deny that the choice of the whole society is not a social choice at all! The problem of social choice is how to design an aggregation device which can yield a commonly accepted social choice when there is a disagreement or conflict. If there is no disagreement or conflict, the problem of social choice simply does not exist. Therefore, unanimity is the 'perfect' way to solve the problem of social choice.

In view of this argument, a unanimity rule, which is a collective decision rule derived from the Pareto relation, is proposed by G. Tullock (1962). "The individualistic theory of the constitution we have been able to develop assigns a central role to a single decision-making rule---that of general consensus or unanimity."[2] The unanimity rule can be expressed as: xPy if and only if xP_iy for all i, where i represents an individual in the society.[3] Undoubtedly, unanimity is a sufficient condition for a social choice. But whether it is a necessary and sufficient condition for a social choice is questionable. Although Arrow has not argued that unanimity should be a desirable condition for a collective choice mechanism, however, some scholars do think so. As it has been noticed by

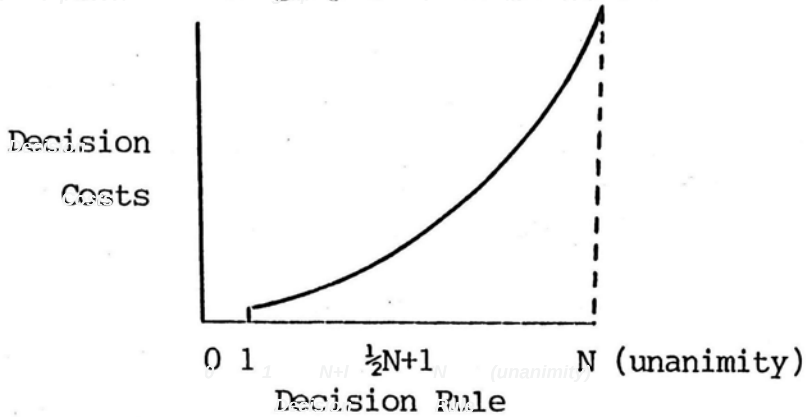
Arrow, "[t]he assumption of...complete agreement among individuals on the ordering of social alternatives, may seem obviously contrary to fact. But, properly interpreted, it is at the basis of a great portion of political philosophy, namely, the idealist school. The fundamental doctrine of the group is that we must distinguish between the individual will, as it exists at any given instant under varying external influences, and the general will, which is supposed to inhere in all and which is the same in all; social morality is based on the latter. This view is expressed in the works of Rousseau, Kant, and T. H. Green, among many others."[4]. And he also adds, "[t]he importance of consensus on ends as part of the process of making judgments on matters of social welfare has been stressed by economists of both the Left and Right persuasions."[5]

Gordon Tullock and James Buchanan (1962) have argued that unanimity rule should be adopted as a collective choice rule. In their book, The Calculus of Consent, they apply certain economic concepts to argue for the unanimity rule. In economic, the costs which are involuntarily imposed to us by the action of others are referred to as negative externalities.[6] Negative externalities, essentially violate individual rights and cause disadvantage to others. Tullock and Buchanan argue that each individual is in the best position to judge whether others' actions cause them a negative externalities. Therefore, they claim that the unanimity rule is best to protect individuals' right. If a certain policy is costly to someone, that individuals can simply veto it. Since every individual has a veto power, only those policies which do not produce external costs to anyone will be implemented. They argue that there is a direct relation between loss of utility and the majority rule. A simply majority rule is the least effective in preventing negative externalities; an absolute majority rule will be more effective, the unanimity rule will guarantee no external costs at all. This argument can be summarized in the table below:



In the figure, the X-axis represents the inclusiveness of the decision rule. The least inclusive decision rule is '1'. This rule says that a policy will be implemented if only a simple individual prefers it. When the decision rule is at 'N', the decision rule is at the extreme inclusive position. This is the unanimity rule, which includes the total number of individuals in the society as a decisive set. The curve slopes down toward the right represents the external costs decline as it approaches unanimity. Tulloak and Buchanan assume that a certain amount of disagreement exists at all time, thus the more inclusive the decision rule is, the lower will be the external costs. The merit of unanimity rule is that it is the best way to guarantee the rights of every individual from the infringement of others since every individual is empowered to veto whatever policies against their own interests.

Tullock and Buchanan, however, realize that it is difficult to achieve unanimity in most situations. When there is disagreement and conflicts, time and effort are necessary to persuade those opposers. Such time and effort are in fact costs for making decision. The persuasion process consume resources and will decrease the utility generated from a policy. Tullock and Buchanan show that this costs can be reduced by replacing the unanimity rule by less inclusive decision rule. The decision cost decreases as the inclusiveness of the decision rule decline. This point can be expressed in graphic form as below:



It can be seen that decision cost curve slopes upward as the inclusiveness of the decision rule towards unanimity. The optimum decision rule will depend on the importance of the potential external costs and the potential decision costs. It can be said that the more homogeneous the society is, the more desirable will it be for the society to adopt a more inclusive decision rule. The existence of decision cost makes it reasonable to choose a decision rule other than unanimity rule. Nevertheless, Tullock and Buchanan are claiming that in a world without decision cost, unanimity rule is desirable, it should be an ideal decision rule for us to pursue. However, it has been pointed out that unanimity rule will easily lead to tyranny of a minority. Since only a unanimously preference is achieved can a policy be implemented, any change from status quo is impossible even there is only a single individual who opposes to the change. It means that the society will prefer status quo when a single individual wants it regardless what the preferences of the rest of the society. It may be another form of dictatorship. Sen has made a comment. "When there is a unanimity of views on some issue, clearly this provides a very satisfactory basis for choice. Difficulties in social choice arise precisely because unanimity does not exist on many questions. What do we do then? One answer is to insist on unanimity for a change, and if there is no such unanimity for any proposed change, then to stick to the status quo. The rule for social choice then can be summed up thus: Given that some prefer an alternative x to the status quo y and no one regards x to be worse than y, x is socially preferred to y; and when this condition is not satisfied, the status quo y is preferred to the other alternative x. This method is one of supreme conservatism. Even a single person opposing a change can block it altogether no matter what everybody else wants."[7] Arrow also says, "Buchanan and Tullock, on the other hand, distinguish between retaining and hanging the status quo most clearly in the following quotation: "We must sharply differentiate between two kinds of decisions: (1) the positive decision that authorizes action for the social group, and (2) the negative decision that effectively blocks action proposed by another group. If a group is empowered to make decisions resulting in positive action by/for the whole group, we shall say that this group effectively 'rules' for the decisions in question. It does not seem meaningful to say that the power to block action constitutes effective 'rule.'... The power of blocking action is not what we normally mean, or should mean, when we speak of 'majority rule' or 'minority rule.'" The asymmetry between action and inaction is closely related to their support of unanimity as the ideal criterion of choice; under such a rule, the status quo is a highly privileged alternative."[8]

I think, although unanimity rule is not desirable in selection of policies, it may be justified in a higher level of selection. That is, the selection of fundamental principles concerning the formation of a society, such as procedures for collective choice and constitutions. This is in a higher order in the sense that they are the 'rules of the game'. These principles must be universally endorsed by every members before they can determine which policies or candidates should be the social choice. The necessary condition for an individual to be a member of a society is that he agrees with the constitution and collective choice rules of that society. In the earliest stage of the formation of a society, those who does not agree to the 'rules of the game' will not join into that society.[9] Therefore, the problem of dictatorship to block the formation of a society simply does not exist in a pre-societal stage. The requirement of procedural unanimity is justified since it is the foundation of a society. Moreover, it might be advisable to propose an unanimously 'acceptable' rather than an unanimously preferred for an alternative. The requirement of unanimously 'preferred' is too demanding. An requirement for unanimously 'acceptable' will be less demanding. It is because an unanimously preferred option implies that an option is acceptable while an acceptable option does not necessarily imply that it is unanimously preferred. One can accept an alternative which is contrary to his own preference. The acceptance can be based on deliberate consideration that the option is the best offer under the circumstances.

In conclusion, I have argued that unanimity is a sufficient condition but not a necessary and sufficient condition for a social choice. Unanimity rule can only be justified in the selection of fundamental principles for a society. Since Arrow's Pareto Principle only requires that a social choice mechanism must guarentee that unanimity is a sufficient condition for a social choice, I think it presents no problem as a requirement for a social choice mechanism.

CHAPTER SIX

NONDICTATORSHIP

This is Arrow's Condition V for a social choice mechanism. It says that no single individual in a society should be permitted to determine the social outcomes regardless of the preferences of others. This condition is obvious, otherwise, the choice cannot be a social choice. However, Strasnick (1976) has suggested that the violation of this condition, in certain situation, is ethically justifiable and there is nothing wrong to allow an individual to be decisive for certain social preference.

Strasnick argues that individuals should not be treated equally in determining a social choice. The preference of the least advantaged individuals should be given priority. This can be justified by Rawls's maximin principle. This principle says that any policy must work to the advantage of the least advantaged members of the society.[1] In a social choice situation, Strasnick claims that the task is to pick up an alternative which would maximize the benefits of the worst-off individual.

Strasnick's argument can be illustrated by an example:

Suppose in a choice setting with two individuals , say individual 1 and 2, and three alternatives x, y, and z, we have the following preference profiles:

Individuals	Alternatives
1	xP_1y, yP_1z, xP_1z;
2	yP_2x, yP_2z, zP_2x.

From the above preference profiles, individual 1 prefers x to y, and individual 2 prefers y to x. There are two exactly opposite preferences, therefore, the society should declare indifferent among x and y. i.e. xIy..................(1)

Since both individual 1 and 2 prefer y to z, by Pareto principle the social preference should be ${}_yP_z$.......(2)

From (1) and (2), and by transitivity, we get the social preference ${}_xP_z$.......................................(3)

Since only individual 1 prefers x to z, while individual 2 prefers z to x, we conclude that individual 1 is almost decisive over x to y. i.e. ${}_xD_y$.................................(4)

Obviously this result violates Arrow's condition of nondictatorship. According to Arrow, this social outcome is undesirable and should not be permitted.

Strasnick, however, argues that the above result is justifiable if information about preferences priority, which is morally relevant information, is taken into consideration for the determination of a social choice.[2] A judgment of preference priority is an interpersonal judgment about the moral status of individual preferences. Moreover, an interpersonal judgment will presumably require some kind of intrapersonal comparison.

In the binary comparisons, x vs. y, and x vs. z ,and y vs. z. Individual 1 orders ${}_xP_{1y}$, ${}_xP_{1z}$, ${}_yP_{1z}$. Individual 2 orders ${}_yP_{2x}$, ${}_zP_{2x}$, ${}_yP_{2z}$. If each individual had a choice of winning on only one binary contest, individual 1 will choose to win on either x vs. y, or x vs. z since his first choice is x. Individual 2 will choose to win on either y vs. x or y vs. z since his first choice is y. Moreover, individual 1 should prefers winning on x vs. z rather than x vs. y, since he has more to lose if z wins than if y wins for z is individual 1's third choice. Similarly, individual 2 should prefer winning in the binary contest y vs. x than in y vs. z for x is his third choice. From these reasoning, we can make an intrapersonal preference priority as follow:

For individual 1, ${}_{x}P_{1z}$ 7 ${}_{x}P_{1y}$ 7 ${}_{y}P_{1z}$.

For individual 2, ${}_{y}P_{2x}$ 7 ${}_{y}P_{2z}$ 7 ${}_{z}P_{2x}$.

(The symbol ' 7 ' reads 'is more important than')

We can see that individual 1 is concerned more with the outcome ${}_{x}P_{z}$ than individual 2 does. Individual 2 has least preference for z over x, hence, if the social choice is ${}_{x}P_{z}$, it will cause little harm to individual 2. The social outcome ${}_{x}P_{z}$, thus can be justified on the ground that it gives individual 1 that he wants most and is the outcome that individual 2 cares least.

We can reach the same argument by interpersonal comparison of utility. For the alternatives x and y, x is individual 1's first choice, y is individual 2's first choice. Strasnick assumes that, in terms of utility, x would provide individual 1 with the same utility as y would provide individual 2. It means that the utility produced by the winning of either one's first choice is the same. Put it in symbols, it means $x_1=y_2$. Strasnick also assumes that the utility of losing is the same for each individual. That is, $y_1=x_2$. The utility received by individual 1 when y is the outcome is the same as the utility for individual 2 when x is the outcome. That is, $y_1=x_2$.

Given these assumptions, we have the following interpersonal comparison:

$$x_1=y_2 \;7\; z_2 \;7\; x_2=y_1 \;7\; z_1.$$

This means that x and y have a higher utility for individual 1 and 2 respectively than z for individual 2, and z has a higher utility for individual 2 than x has. Moreover, the utility of x for individual 2 is equal to the utility of y for individual 1, and the utility of y for individual 1 has greater utility than z has. From these set of relations and by transitivity, we have $x_1 > z_2 > x_2 > z_1$. We can see that individual 2 receives from x, his lowest preference, is greater than the utility which individual 1 receives from z, his lowest preference. By having the social choice to be x over z is less harmful than the social choice to be z over x. Individual 1 has more to gain than individual 2 if his preference is satisfied, and more to lose if it is not . Since we are choosing that preference which maximizes the utility of the worst-off individual , by Rawls's maximin principle, the social choice ${}_xP_z$, is ethically justifiable.

I think, Strasnick's preference priority method at most can apply to situations where individuals are choosing among policies which have distributive effects to them. However, when individuals are choosing among candidates for a position where the benefits are going to the candidate, not to the choosers, the preference priority procedure would be inapplicable.

Moreover, I think Strasnick's major problem is that if we allow an individual is decisive in one pair of alternatives, he will then be decisive in all pairs of alternatives as we have proved previously. It means that the individual not only is decisive in the pair he would otherwise be worst-off, but also is decisive in all other pairs. This consequence is unacceptable.

I think the violation of the nondictatorship requirement will make the choice thus obtained the choice of an individual only, not a social choice at all.

CHAPTER SEVEN

THE NOTION OF RATIONALITY

As I have already mentioned, Arrow's conception of rationality is the same at the individual and collective level. Both of which has to fulfill Axiom I and Axiom II. "[I]t will be assumed that individuals are rational, by which is meant that the ordering relations R_i satisfy Axiom I and Axiom II. The problem will be to construct an ordering relation for society as a whole that will also reflect rational choice-making so that R may also be assumed to satisfy Axiom I and Axiom II."[1] Arrow's requirement that individual preferences should be connected and transitive is referred to as his condition of individual rationality. His requirement that group preferences is connected and transitive is referred to as his condition of collective rationality.

The question now I try to raise is that: Why should we accept Arrow's definition of rationality? Does the notion of Arrow's rationality capture the ordinary sense of this term?

In ordinary sense, to act rationally means to give reasons to support an action. As has been suggested by Peter C. Ordeshook, "In the case of human action, to give a reason is to demonstrate that the actors acted as if they were seeking to achieve a particular goal."[2] Thus, rationality can mean the selection of means to achieve an end. An individual is rational if he selects the best means (the most effective, efficient, etc. method) available to him to achieve his goal. A judgment of rationality or irrationality is thus a judgment about the causal relation between an action considered as a means and a given end.

But if there are more than one end to achieve, and these ends may or may not be compatible with others, then we have to make a choice among the ends. Before making a choice about means, compatibility among ends must be ensured since conflicting ends cannot be attained simultaneously. This is the requirement of consistency among the ends. Even in the set of compatible ends, we may have to set a priority in case we have to abandon some ends for a practical reason. In this sense, a judgment of rationality and irrationality is a judgment about the logical relation (consistency, ranking) among alternatives.[3]

From the above discussion, we notice that the conception of rationality in ordinary sense has at least two meanings:

First, the selection of best means to achieve desired ends;
Second, the consistency among the ends.

Arrow's definition of rationality is incomplete. He only restricts his definition in a very narrow sense that a choice is rational if it is consistent. (The Axiom I and II is exactly the condition of consistency. We shall see this point later.)

However, I shall examine both the individual and the group level to see whether the same notion of rationality may apply. Moreover, I shall also evaluate the condition of consistency, to see if it is a necessary, sufficient or necessary and sufficient condition for a rational choice.

The assessment will be divided into two parts. The individual level and the group level.

§7.1 Rationality Principle at the Individual Level

The Axiom of connectivity allows an individual either prefers one to the other or be indifferent between them. For two alternatives x and y, there are only three logically possible relations. The relations are ${}_xP_y$, ${}_xI_y$, and ${}_yR_x$. Therefore, the classification of ${}_xR_y$ and ${}_yR_x$ is exhaustive. For there is a choice situation, the preference must fall in either one of the categories. For a classification, there is no question of its truth or false. It only exists the question of applicability. I would show that, the Axiom is not applicable in some cases.

Supposing I want to buy a car, among all alternatives, I pick out two cars, say, x and y, for my further consideration. I evaluate them with multi criteria such as price, color, brand name, style and extra. With respect to price, I prefer x to y. Considering color, I prefer y to x; but I like car x's style rather than y's. However, I cannot make up my mind on which car I should purchase. In this situation, I cannot said to be indifferent among x and y.[4]

Now I have been thinking about the choice for more than one day and still cannot make up my mind. In order to solve the problem and save my time, I flip a coin to determine which car I should buy. Since I resort to a randomizing method to make choice. It seems that I am indifferent between car x and y. Although finally I decide to choose car x. Does it mean that I prefer x to y? According to Axiom I, if I finally choose x rather than y, I am presumed to hold that x is at least as good as y. i.e. ${}_xR_y$. How does the axiom help us to distinguish the situation of choosing x to y; and , the situation in which x and y is indifferent but x is chosen because I have to choose something, and the situation in which I have preferences among x and y with multi criteria but seems to be indifferent since I cannot make up my mind? I think the axiom has little use.

The second axiom proposes by Arrow is the principle of transitivity. It says that if x is at least as good as y and y is at least as good as z, then, we will infer that x is at least as good as z. The axiom applies to a choice over three alternatives.

In a logical system, a proposition is an axiom if and only if it is a fundamental hypothesis, and it cannot be derived from other propositions in the system.

Arrow claims that the principle of transitivity is an axiom. This means that Axiom II is independent, and fundamental. However, I shall try to show that it is not the case. The principle of transitivity can be derived from Axiom I alone. It means that the principle of transitivity is neither independent nor fundamental; it is only a theorem. Hence, the 'Axiom II' is not an axiom at all and should be considered as redundant.

Now I try to prove the following proposition by the method of contradiction.[5]

For all x, y, and z, ${}_xR_y$ and ${}_yR_z$ imply ${}_xR_z$.
This is exactly the principle of transitivity.

We first postulate the following definition:
"Definition 1: ${}_xP_y$ is defined to mean not ${}_yR_x$."
The statement "${}_xP_y$" is read "x is preferred to y."[6]

Suppose we have the following preferences:

1. ${}_xR_y$ (Given)
2. ${}_yR_z$ (Given)
3. ${}_zR_x$ (Assumption)

Assume conditions 1, 2 and 3 are consistent. From a consistent preferences profile, we can make a choice among three alternatives, say, x, y and z.

Step 1

Now supposing that an individual 1 chooses alternatives x. It will mean he rejects y and z. Hence we have the preference profile: not ${}_{y}R_{x}$ and not ${}_{z}R_{x}$.

However, this preference profile violates condition 3 that ${}_{z}R_{x}$.

Hence, individual 1 cannot choose alternative x.........(1)

Step 2

Assuming that individual 1 chooses alternatives y and rejects x and z. This means that: not ${}_{x}R_{y}$ and not ${}_{z}R_{y}$.

This preference profile is a violation of condition 1 that ${}_{x}R_{y}$.

Therefore, individual 1 cannot choose alternative y.....(2)

Step 3

Now considering that individual 1 chooses alternative z and rejects x and y. i.e. not ${}_{x}R_{z}$ and not ${}_{y}R_{z}$.

However, it violates condition 2 that ${}_{y}R_{z}$.

Hence, individual 1 cannot choose alternatives z........(3)

Step 4

From the result (1), (2) and (3) above, we conclude that the individual 1 cannot choose among the three alternatives x, y and

z. It means that our assumption that the preferences profile: $_xR_y$, $_yR_z$ and $_zR_x$ are consistent is false. That is, condition 1, 2 and 3 are inconsistent.

Among the three conditions, condition 1 and 2 are given. Therefore, our assumption that $_zR_x$ holds, must be false. i.e. not $_zR_x$. From definition 1, it is equivalent to $_xP_z$.

We thus obtain $_xR_y$ and $_yR_z$ imply $_xP_z$....................(4)

From definition 1 and Axiom I, $_xP_z$ imply $_xR_z$............(5)

Combining the results of (4) and (5), we get:
$_xR_y$ and $_yR_z$ imply $_xR_z$.

We thus have proved the principle of transitivity can be derived from the Axiom I alone.

We can see that Axiom I is the sufficient condition for the principle of transitivity. Rejection of the principle of transitivity will also mean the abandon of the Axiom I. Therefore, the attack on the principle of transitivity will mean the attack against both the Axiom I and 'Axiom II'. In other words, Arrow's conception of rationality is put in doubt.

C. Dykes [7] has distinguished three different types of preferences, which can facilitate us to illustrate that the violation of transitivity, in some cases, is not irrational.

The three types of preferences are:

(1) A pure preference: A preference which has no reason to support.

(2) A reasoned preference: A preference which is supported by reasons.

(3) A all-things-considered preferences: A preference which is arrived after a deliberated weighting and calculation of all relevant factors and consequences.

With the help of C. Dyke's distinction between different types of preferences, I shall try to illustrate that in some cases, an intransitive preference is not necessarily irrational.

Case 1: In pure preference.

People may choose simply out of an impulse. In the presence of alternatives x and y, the individual chooses x out of impulse 1. While comparing y and z, that individual prefers y out of impulse 2. When facing x and z, the individual selects z out of impulse 3. In the presence of all three alternatives, impulse 1, 2 and 3 simultaneously appear. The three impulses may be disconnected and independent. Therefore, the preferences in the previous three choice settings are disconnected, and transitivity becomes impossible and meaningless.

Case 2: In a reasoned preference.

Suppose I am asked to compare three alternatives, x, y and z. First I have reason to prefer x to y and y to z. However, in the second round of comparison, I change my mind and rank z over x. Although the preferences I make in the first and second round of comparison are inconsistent. I am not irrational since each preference I make is supported by reason.

Now supposing the time duration for the first round and the second round comparison is quite short that we can consider the comparisons performed simultaneously. It means that after I make a preference for x to y and y over z, within a split of second, I change my mind and rank z over x.

The above intransitivity should not be viewed as irrational. Since every choice is a rational choice in the sense that they are supported by reason.

We should notice that in reality, all choices must be taken within certain time. However, the principle of transitivity says nothing about the time dimension. The above two cases illustrate one important point, that the principle of transitivity only applies when the state of mind of an individual is stable, is using the same criteria for the ranking of alternatives in a certain duration of time in which choice actions are taken.

Intransitivity can also occur in the domain of probabilistic reasoning. Suppose there are three perfectly fair dice. Die A's six faces are 6, 6, 6, 2, 2, 2; Die B's are 5, 5, 4, 3, 3, 2; and Die C's are 6, 4, 3, 3, 3, 3.[8]

It can be calculated that rolling two die at a time, say die A and die B, on any given roll, the probability of A beating B is 6/12, the probability of B beating A is 5/12. We can see that the probability of A beats B is higher than B beats A. Therefore, it is rational to bet on A against B. i,e. $_AP_B$.

When rolling die B and die C, the probability of B beating C is 14/36, the probability of C beating B is 13/36. Hence, it is rational to bet on B against C. i.e. $_BP_C$.

When rolling die C and A, the probability of C beating A is 6/12, the probability of A beating C is 5/12. Undoubtedly it is rational to bet on C against A. i.e. $_CP_A$.

In the long run, for a rational gambler, it will be wise to bet on A against B, B against C and C against A. Again, the principle of transitivity does not hold.

From the above example, we can notice that, in face of uncertainty about the outcome, in some cases transitivity does not hold. Therefore, the violation of the transitivity in some situations need not be considered as irrational.

A pyschologist, A. Tversky has suggested that we should defined the notion of transitivity in terms of probability since individuals are not perfectly consistent in their choices when faced with repeated choices.[9] In A. Tversky's probabilistic version of transitive preference, which he terms as weak stochastic transitivity, WST, the validity of the transitivity principle is conditional. That is, the preferences must first fulfill certain condition before we can draw a transitive preference from them. I think his probabilistic version of transitive preference can also be applied in choice setting involved probabilistic calculation. I shall reformulate his version and present in the following way: Suppose P(x, y) is the probability of x beating y when both x and y are present, and P(y, x) is the probability of y beating x when both x and y are present. Under the condition $P(x, y) + P(y, x) = 1$ holds (i.e. either x beats y or y beats x). Preference may be defined by $_xR_y$ if and only if $P(x, y) \geqslant \frac{1}{2}$, for all variables x and y.

For three alternatives, x, y, and z;
$P(x,y) \geqslant \frac{1}{2}$ and $P(y, z) \geqslant \frac{1}{2}$ imply $P(x, z) \geqslant \frac{1}{2}$ if the conditions $P(x, y) + P(y, x) = 1$, $P(y, z) + P(z, y) = 1$, and $P(x, z) + P(z, x) = 1$, hold.
This is the notion of weak stochastic transitivity.

Since preference is defined by $_xR_y$ if $P(x, y) \geqslant \frac{1}{2}$, for all variables x and y. Therefore, the principle of transitivity can be restated as:

For all x, y, z, $_xR_y$ and $_yR_z$ imply $_xR_z$ if the condition of connectivity holds. (Since $P(x, z) + P(z, x) = 1$ implies either $P(x, y) \geqslant \frac{1}{2}$ or $P(y, x) \geqslant \frac{1}{2}$. It means either $_xR_y$ or $_yR_x$ holds. This is exactly the condition of Axiom I.)

Comparing this reformulated version of the principle of transitivity with the one originally proposed by Arrow, which he suggests as the Axiom II, we can notice that the truthness of Axiom I implies the truthness of the principle of transitivity. Therefore, Axiom I is the sufficient condition for the principle of transitivity. Again, this is another evident to prove that 'Axiom II' is a theorem only.

In the above case, we can notice that the condition $P(x, y) + P(y, x) = 1$ does not hold. For die A and B and C, $P(W_{AB})+P(W_{BA})=0.916$; $P(W_{BC})+P(W_{CB})=0.749$; $P(W_{CA})+P(W_{AC})=0.920$ Therefore, no transitive preference can be obtained. (Where '$P(W_{AB})$' reads 'the probability of A beating B.')

Recalling the fact that the principle of transitivity is the necessary condition for connectivity (that is the requirement of consistency), thus the violation of the principle of transitivity means the violation of the condition of consistency. We conclude that consistency among preferences is not a necessary condition for a choice to be rational.

In view of means-end efficiency as the notion of rationality, the condition of consistency is obviously not a sufficient condition for a rational choice.

From the above analysis we thus conclude that the condition of consistency is neither a necessary nor a sufficient condition for a rational choice at the individual level.

§7.2 Rationality Principle at the Group Level

According to Arrow, collective rationality bears the same notion of individual rationality. Therefore, I suppose the same conception can be used for us to understand what collective rationality really means as suggested by Arrow.

If we use the same notion of rationality at the individual level to define the rationality at the group level, we may define collective rationality as :

1. A social choice is rational if the society selects the best means or choice to achieve certain social ends.
2. A social choice is rational if choices of the society are consistent. (here the term society is used in its broad sense, it means, any organized group of people.[10])

We should notice that society is not an organism cable of making its own choice. We say a society has a choice is an anthropomorphic expression. The choice of a society is merely the outcome of a social choice mechanism with individual preferences as the input. Can we assess the means-end relations of a group to see whether a social choice is rational or not, just the same way as at the individual level? First of all, in order to find the best means, we have to define what the goal is. An individual could have a clear mind of what he wants so that he can decide which choice should be made for the best realization of his end. It presents no difficulty for an individual to define his end. However, for a society, the social ends are not known.

As Arrow is an individualist, he will insist that a social choice should and only should be based on individual preferences. Moreover, he also stipulates the condition of citizens' sovereignty as a 'reasonable condition' for an aggregation device.

Clearly Arrow will deny any apriori social end which is unrelated to individual's preferences. The 'social end', unlike the end of an individual , cannot be known in advance. It is possible of being known only after all the preferences of the individuals of the society have been processed through a social choice mechanism. Therefore, it is meaningless to view the collective rationality from the means-end efficiency perspective.

Arrow defines the notion of collective rationality by Axiom I and II. In fact, the two axioms uphold the condition of consistency. Thus collective rationality can be interpreted as the consistency among a set of social choices which are based on the same set of individual orderings of a society. Arrow's interest is to find a social choice mechanism which can guarantee consistency among social orderings. Therefore, "[t]he problem will be to construct an ordering relation for society as a whole that will also reflect rational choice-making so that R may also be assumed to satisfy Axiom I and II."[11] After having clarified the concept of collective rationality, the remaining question is that: Why should the condition of consistency be desirable for social orderings? Arrow seems to think that it is obvious and need not provide any answer. However, I shall argue that the condition of consistency is not necessary. Moreover, this requirement is the violation of values proposed in a pluralistic society.

Peter C. Fishburn (1970) has suggested that social transitivity is not necessarily a desirable condition for a collective choice mechanism. He shows that in certain situations, a social outcome which violates Arrow's principle of transitivity at the group level appears to be acceptable and reasonable.[12]

Let us consider Fishburn's example.

Suppose there are three candidates x, y and z competing in an election; and there are 21 voters. The preferences profile for the 21 voters are listed below:

VOTERS	RANK ORDER OF CANDIDATES		
	1st	2nd	3rd
1 to 10	x	z	y
11 to 20	y	x	z
21	z	(x and y is indifferent)	

We can see that candidate x receives ten first-place votes, eleven second-place votes. Candidate y receives ten first-place votes, one second-place votes and ten third-place votes. Candidate z receives only one first-place vote, ten second-place votes and ten third-place votes. Among the voters, ten have the preference ordering $_xP_z$ and $_zP_y$; ten have $_yP_x$ and $_xP_z$, and one has $_zP_x$ and $_xI_y$.

Fishburn argues that it is reasonable for candidate x to be the winner even though that candidate does not have a majority of the first-place votes. It is because candidate x has ten first-place votes and eleven second-place votes. Candidate y only has ten first-place votes and one second-place vote. For candidate z, he only receives one first-place vote and ten second-place votes.

In the above election, although there is no Condorcet winner since no candidates beats every others in a pairwise contest. If we replace the Condorcet criterion by a less stringent one: the extension condition, there will be a winner. The extension condition says that a candidate is a winner if he defeats most of the other candidates and tie with the rest in a pairwise contest. According to extension condition, candidate x should be the winner since he defeats z and ties y, and is the only candidate beats most and tie the rest of other candidates.

On the assumption that candidate x should be the winner in the election, let us consider whether social transitivity holds in this case.

From the given preference profile of voters, we can get the following social orderings:

(i) Since 20 voters prefer x to z, the social preference should be ${}_xP_z$.

(ii) Since 11 voters prefer z to y, the social preference should be ${}_zP_y$.

(iii) Since x ties with y, the social orderings should be ${}_xI_y$.

From (ii) and (iii) and by the principle of social transitivity, we have ${}_zP_x$. However this contradicts to (i) that the social preference is ${}_xP_z$.

Thus we can see that social transitivity does not hold even though we have reson to consider candidate x to be the winner in this case. This means that consistency is not a necessary condition for a social choice mechanism.

The requirement of consistency among social orderings also violates some values upheld by a pluralistic society.

For example, in a populistic democratic society as defined by Robert A. Dahl.

> " DEFINITION 1: An organization is democratic if and only if the process of arriving at government policy is compatible with the condition of popular sovereignty and the condition of political equality.
>
> DEFINITION 2: The condition of popular sovereignty is satisfied if and only if it is the case that whenever

policy choices are perceived to exist, the alternative selected and enforced as governmental policy is the alternative most preferred by the members.

DEFINITION 3: The condition of political equality is satisfied if and only if control over governmental decisions is so shared that, whenever policy alternatives are perceived to exist, in the choice of the alternative to be enforced as government policy, the preference of each member is assigned an equal value.

DEFINITION 4: THE RULE: The principle of majority rule prescribes that in choosing among alternatives, the alternative preferred by the greater number is selected. That is, given two or more alternatives x, y, etc., in order for x to be government policy it is a necessary and sufficient condition that the number who prefer x to any alternative is greater than the number who prefer any single alternative to x." [13]

In a pluralistic society, individuals are allowed to have different values and different preferences. Their preferences are assigned an equal value. It is permissible for every individuals in that society to have exactly different preference profile. Under a populistic society, citizens may divide into equal groups and prefer differently among alternatives. Suppose there are three candidates x, y, and z, and the citizens divide into three equal groups as follows:

Group A prefers x to y, and y to z, and x to z.
Group B prefers y to z, and z to x, and y to x.
Group C prefers z to x, and x to y, and z to y.

We can notice that this is the preference profiles for a voting paradox. Thus under a populistic democratic society, an inconsistent social orderings can be obtained.

The inconsistent social outcomes may just be a reflection of the diversified and equal divisions of preferences of citizens. Every citizen can make their free choices according to their own tastes. Therefore, inconsistent social orderings are permissible. If the inconsistent social outcomes are termed as 'irrational', then , it is equivalent to say that diversity and equality of values of preferences are undesirable and unacceptable.

Arrow, as an individualist, intends to construct a social welfare function to fulfill certain conditions which he think to be reasonable. If we look closer into these conditions, we can find that they express the very belief of populistic democracy.[14] Thus, Arrow's individualist outlook should be in line with inconsistent social orderings. Arrow's position, on the one hand, is an individualist while on the other hand, suggests that only consistent social orderings can be said to be collectively rational, appears to be self contradictory.

As has been commented by Amartya K. Sen, "Arrow's impossibility theorem is precisely a result of demanding social orderings (the requirement of consistency) as opposed to choice functions."[15] R. A. Dahl has also added, "[b]ecause Arrow, assumes "transitivity of collective choice" as a criterion of rational social action, it is worth noting that under almost any theory of democratic politics, and certainly under the one in question here (the populistic democracy), the requirement of transitivity would be irrational in a great many types of collective choices....clearly it would lead to irrational results in a democracy to require transitivity in collective choices."[16]

Although we can argue that violation of social transitivity is not unreasonable, and inconsistency among social orderings should be permitted in view of upholding the values of populistic democracy. However, there remains an important practical problem unsolved. In case of inconsistent social orderings arise, what should be the social choice? I think some inconsistency-breaking mechanism should be introduced. But how should we design this inconsistency-breaking mechanism? We may, however, get some ideas from the tie-breaking mechanism in rules of meeting.

"A tie vote on a motion means that the same number of members has voted in the affirmative as in the negative. Since a majority vote, or more than half of the legal votes cast, is required to pass a motion, an equal or tie vote means that the motion is lost because it has failed to receive a majority vote. A tie vote on a motion is not a deadlock vote that must be resolved; it is simply not a majority vote and the motion is lost. A tie vote that constitutes a deadlock that must be resolved can occur only when two or more candidates, or two or more alternative propositions, are being voted on at the same time and two or more of them received the same number of votes. Then no candidate has been elected, and no proposal has been adopted. Such a tie vote results in a deadlock, and the vote must be retaken until the tie is resolved by voting or by some other method which the assembly may choose."[17]

In some committee, whenever a tie is observed, the chairman in the committee is entitled to have a casting votes to break the tie. "By "casting vote"...what is meant is a second vote exercisable by the chairman of a meeting in addition to his own vote as a member."[18]

In a voting paradox, there is an evenly distributed votes among different preferences profile, if we regard it as equal groups of individuals voting for exactly different motion, then the situation is just the same as in a tie voting.[19] Therefore, whenever there is a paradox of voting, we can simply declare all the alternatives are socially indifferent to each other, and follow tie-breaking mechanism to break the paradoxical situation.

In the case of tie voting, we can either re-vote for those motions until the tie is resolved; or we can entitle the chairman to have a casting vote. The second measure seems preferable since it is more time saving. However, this measure appears to violate the condition of political equality. But, is the infringement negligible? In order to find out the answer, let us consider the situation below:

A committee consists of four members, say, x, y, z, and w. Moreover, x is the chairman of the committee. In the committee, each member has one vote. Whenever a tie is noted, the chairman will be empowered to break the tie. The question is: How much voting power do the chairman have more than other committee members? Voting power may be defined as the chance that a given individual's vote will be crucial to the decision voted by the body. We use shapley-Shubik index to measure the voting power of every individuals in the committee.

In an abstract setting, we would not have apriori knowledge about possible orders of coalition formation. Shapley and Shubik propose that to measure abstract voting power, we should consider all orders equally likely. A member is said to be <u>pivotal</u>, if the losing coalition will become winning only when that member joins it. It means that the pivotal member holds the power. Supposing a motion is put , y strongly supports the motion, w also support the motion. Obviously y and w will join a coalition in support of the motion. At this point, the motion would still lose. The losing coalition will be able to win only if it can gain the support from the third member. Hence the third member has the crucial power in this situation. If z strongly opposes to the motion and x is indifferent, then x is the pivotal member.

In the following analysis of voting power, we assume that the probability of all orders of coalition formation are equally likely.

For a committee with four members, there are 4! (i.e. 24) possible ways of ordering. The ordering is arranged in descending order of the degree of support. For example, x $\underline{y}$ z w, it means x strongly support the motion, y is the second most support the motion, z is indifferent while w strongly opposes the motion. The pivotal member has been underlined in the ordering. The 24 possible ways of ordering are:

x $\underline{y}$ z w	x $\underline{w}$ y z	y z $\underline{x}$ w	z $\underline{x}$ y w	z w $\underline{x}$ y	w y $\underline{x}$ z
x $\underline{y}$ w z	x $\underline{w}$ z y	y z $\underline{w}$ x	z $\underline{x}$ w y	z w $\underline{y}$ x	w y $\underline{z}$ x
x $\underline{z}$ y w	y $\underline{x}$ z w	y w $\underline{x}$ z	z y $\underline{x}$ w	w $\underline{x}$ y z	w z $\underline{x}$ y
x $\underline{z}$ w y	y $\underline{x}$ w z	y w $\underline{z}$ x	z y $\underline{w}$ x	w $\underline{x}$ z y	w z $\underline{y}$ x

The Shapley-Shubik power indices of the members are thus 12 out of 24 for x, 4 out of 24 for y, z, and w:

That is, (12/24, 4/24, 4/24, 4/24)
x y z w

We can notice that if the chairman is entitled to have a casting vote, he will have three times as much power as each of the other committee members. This measure severely infringes the condition of political equality.

Therefore, a re-voting method may be more reasonable for the inconsistency-breaking mechanism. The paradox of voting may still occur in the consecutive voting. However, after bargaining and lobbying among the members, I think the occurrence of the paradox will be practically insignificant. However, some may argue that the paradox may still persist in several consecutive re-voting and create a practical problem. In this situation, what should we do?

The above situation is only logically possible, however, is implausible. But if it is really the case, we may employ some randomizing methods as a last resort to dissolve the paradox, say, by throwing a fair die or drawing of lots. Some may challenge these methods as unreasonable and arbitrary. In responding to this attack, I shall first recall the very notion of rational choice.

A choice is said to be rational if it is guided by reason or principles. Whenever a choice is guided by principles or reason, it is a rational choice. In normal situations, social choices are guided by the principle of majority. They are principle guided choices and hence are collective rational. However, in the case of inconsistency among social orderings, the majority principle should have to be set aside and other means are employed. As I have argued previously that the alternatives for a society should be declared indifferent to each other when paradox occurs for we have no reason to say any alternative should be preferred. All alternatives thus should be assigned an equal value for the society, and each alternative should have an equal probability to be the choice of the society. A randomizing method is used because it uphold both the condition of political equality and the principle of fairness. They are the fundamental principles we have to recognize. Therefore, it is not unreasonable to use a randomizing method to dissolve a paradox. Moreover, there is in fact a rule that governed the social outcome. The decision rule is that: A choice is the choice of the society if and only if it is obtained by a perfectly fair randomizing method. It can hardly be denied that the choice from a randomizing method is also a rule governed choice.

In conclusion, I suggest in this chapter that inconsistency of social orderings is permissible. Whenever inconsistent social orderings are observed, re-voting is a reasonable measure to be taken. As a last resort, a randomizing method can be introduced. Under a repeated paradoxical situation, a randomizing method is well justified and acceptable. Therefore, inconsistent social choices are ethically justifiable by populistic democratic theory and also practically dissolvable by a re-voting and randomizing method. The condition of consistency, which I have already argued, is not a necessary condition for a social choice mechanism.

SECTION III

ELECTORAL SYSTEMS

CONDITIONS FOR ELECTORAL SYSTEMS

GENERAL SURVEY OF ELECTORAL SYSTEMS

CHAPTER EIGHT

CONDITIONS FOR ELECTORAL SYSTEMS

In previous chapters I have assessed the bases of reasons for Arrow's conditions as a social choice rules. In the following chapters, I want to explore a wide range of electoral systems currently practised in democratic countries to see whether these systems satisfy the minimum requirements proposed by Arrow. The analysis performed may be rather formal, therefore, some mathematical notations and more precise definition for Arrow's conditions will be given as a preparation for our further discussions.

§8.1 Preliminary Notations and Definitions

First we set forth the following notations:

1. X is the universal set of alternatives, which is a nonempty finite set.
2. S is a nonempty subset of X, to be interpreted as the set of alternatives that are currently available, is called an opportunity set or the choice environment.
3. R_i is an ordering of X held by individual i
4. R_i^S is the relation R_i over a subset of X.
5. R is a social ordering on X.
6. R^S is the relation R over a subset of X.
7. P_i is the strict relation corresponding to R_i, xP_iy to be called x is strictly preferred to y by individual i.
8. P is the strict relation corresponding to R, xPy to be called x is strictly preferred to y by the society.
9. u is a preference profile for a set of individuals, i.e. $u=[R_1, R_2, \ldots, R_n]$.
12. F is the social choice rule which governs the selection of a social choice mechanism. e.g. Arrow's conditions.
13. C is the social choice mechanism, which is a function of a social choice rule. i.e. C=F(u) e.g. electoral systems

14. $C(u, S)$ is the choice set of S with respect to u by the social choice mechanism C.
15. N is the number of individuals in the choice environment.
16. $N({}_xR_{iy})$ is the number of individuals i N for whom x is being preferred to y.
17. V is the total votes returned in any constituency in an election.
18. t is any single individual or party's share of the total vote.
19. M is the total seats assigned to any constituency.

§8.2 Social Choice Rules for the Assessement of Electoral Systems: A Reformulation of Arrow's Conditions

In the context of Arrow's impossibility theorem, several conditions are being put as reasonable social choice rules for any social choice mechanism. They may now be summarized in the following:

(A_1) There are at least three individuals in N i.e. $N \geqslant 3$.

(A_2) There are at least three alternatives in X i.e. $X \geqslant 3$.

(A_3) R is connective: ${}_xR_y$ or ${}_yR_x$ (or both) for all $x, y \in X$.

(A_4) R is transitive: for all $x, y, z \in X$, if both ${}_xR_y$ and ${}_yR_z$, then ${}_xR_z$.

(A_5) Condition U (Unrestricted Domain): This condition means society's choice must be obtainable from any pattern or combination of individual orderings. No patterns or combination of individual orderings can be excluded to yield a social choice. In other words, the domain of the social choice function F shall include all possible combinations of individual orderings in all possible environment within the universal set of alternatives X. This condition has two implications: (1) All logically possible profiles of preference orderings on X are admissible. (2) All logically possible combination of alternatives in X are admissible.

(A_6) Condition I (Independence of Irrelevant Alternatives): For $u=[R_1, R_2, \ldots, R_n]$ and $u'=[R_1', R_2', \ldots, R_n']$, if $R_i^S=R_i'^S$ then $C(u, S)=C(u', S)$.

(A_7) Condition P (Pareto Principle): For any pair x, $y \in X$, if $_xP_{iy}$ for all individuals i, then $_xP_y$.

(A_8) Condition N (nonditatorship Principle): There is no individual i in the society such that: if $_xP_{iy}$ then $_xP_y$ for all x, $y \in X$.[1]

Arrow's conditions are, in fact, a set of rules for a social preference function. (In Arrow's own term, they are conditions for a social welfare function.) The chosen set for a social preference function is a social ordering. This is, however, different from that of social choice function. For a social choice function C, which assigns to each u and each S, a nonempty subset, C(u, S) of S is chosen. The chosen set, C(u, S), is a set of winners, in which ordering is not necessary.

Electoral system, however, is a social choice function, by which for each voters' preference profile u, and a given set of candidates S, a winner (or winners) in S is/are obtained. In a single constituency, more than one winner might be declared as a result of ties. This situation can be resolved by some built in tie-breaking mechanism in an electoral system.

A social choice function, of course, should also obey certain elementary rules. To serve our purpose to select an acceptable electoral system, or exactly it may be called a social choice function, Arrow's conditions need to be redefined so that we can identify social choice function conforming to those criteria.

As we have discussed earlier, (A_3) entails (A_4), and (A_4) is in fact the requirement of consistency. Therefore we can put a social choice rule, Rule C as the rule of consistency for any social choice mechanism as:

Rule C: A social choice mechanism C is consistent if and only if: for all $x, y \in X$, if $x \in C(u, [x, y])$ and $y \in C(u, [y, z])$ then $x \in C(u, [x, z])$. For a staging social choice mechanism, the consistency condition can be interpreted as: For $x \in S' \subset S \subset X$, if $x \in C(u, S)$ then $x \in C(u, S')$ (It means that if an alternative is chosen in a larger choice environment, he will also be chosen in a smaller choice environment.)[2] The reason is: If we have $x \in C(u, [x, y, z])$, then we would conclude that x is at least as good as x and y. But if $x \notin C(u, [x, y])$, it will mean that x is not at least as good as y. This is a contradiction. Therefore, x must be an element in $C(u, [x, y])$

(A_1), (A_2) and (A_5) can combine together to yield a requirement on the domain of a social choice mechanism. That is, the social choice mechanism should respond to all logically possible combinations of individual orderings in all possible environments for individuals and alternatives more than three.

This rule can be called Rule U (Rule of Unrestricted Domain):

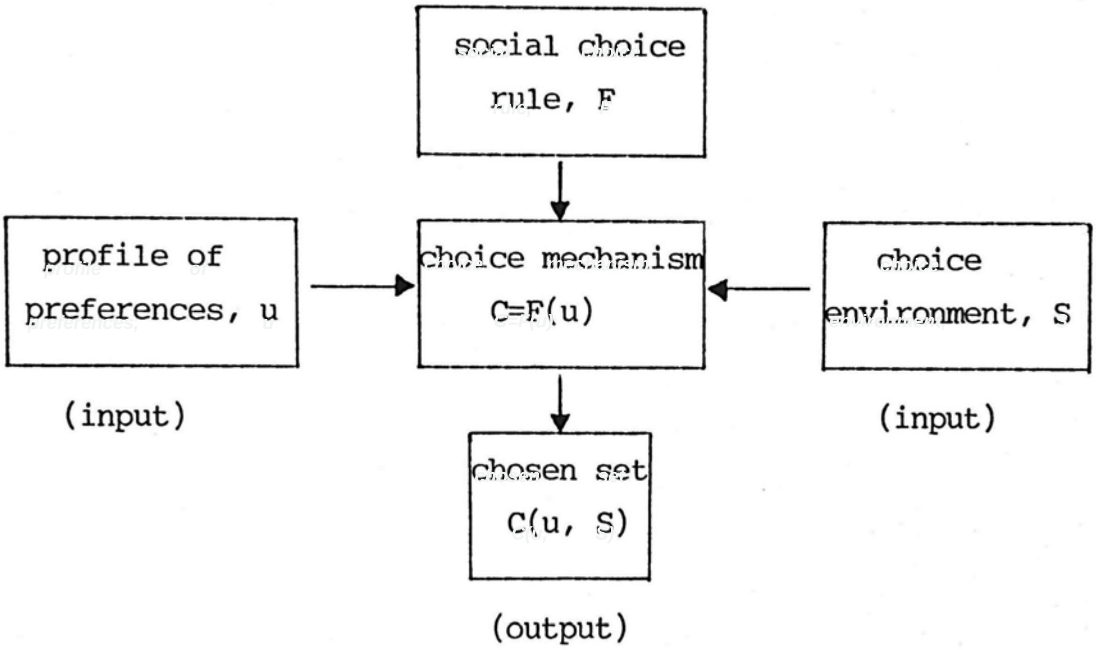
Rule U: i) $N \geqslant 3$

ii) $X \geqslant 3$

iii) the social choice mechanism has a domain as all logically possible preferences of orderings on X.

iv) the social choice mechanism has a domain as all logically possible combinations of alternatives in X.

It should be noticed that a social choice mechanism is a function of a social choice rule, and the output of a social choice mechanism is determined by two variables, namely: the preference profile for a set of individuals (u), and the choice environment (S). This can be illustrated in the following figure:



A social choice mechanism is said to be independent of irrelevant alternatives if candidate x in S is declared as a winner on the basis of comparisons with other members in S, x's status as winner should not be affected by alternatives outside the given choice environment S. Thus, we have:

Rule I: For $u=[R_1, R_2, \ldots, R_n]$ and $u'=[R_1', \ldots R_n']$, and for all $x \in S \subset X$, if $R_i^S = R_i'^S$ and $C(u, S)=[x]$, then $C(u', S)=[x]$.

(It should be noticed that the chosen set $C(u, S)$, is interpreted as a set of winners, not a social ordering.)

Arrow's condition P is restricted in a pairwise comparision between alternatives, it says, if $_xP_{iy}$ for all individuals i then $_xP_y$. It does not necessarily means that x is an element in a chosen set for the society, since it may be the case that everyone agrees y is the last and x is the second last to be preferred. Moreover, maybe there is an alternative, say, z, in S that everyone prefers to x. Therefore, even though everyone in the society prefers x to y, it does not necessarily mean that x should be in the chosen set. It is true only when the choice environment containing two alternatives. However, it is reasonable to conclude that y should not be chosen if x is available. In view of this interpretation, a social choice mechanism satisfies Pareto Principle if everyone in the society prefers x to y, then it does not choose y when x is available.

Rule P: In a preference profile u, if $_xP_iy$ for all individuals i, then $y \notin C(u, S)$ when $x \in S$.

From Arrow's Condition N, we know that an individual i is an dictator if $_xP_iy$ then $_xP_y$. But it does not necessarily means that x is being chosen since in the social ordering ranking, y may be ranked as the last and x is the second last. Both x and y may not be chosen by the society. However, if y is chosen by the society, x should also be included in the chosen set. Since only individual i prefers x, therefore i is a dictator. This is what Condition N prevents. Hence, for a social choice mechanism satisfies Condition N, it must not choose x when y is in the chosen set provided only a single individual i prefers x to y.

Rule N: There is no single individual i in the society such that $_xP_iy$ and $y \in C(u, S)$ implies $x \in C(u, S)$, for $x, y \in S$. It is equivalent to say that: If $_xP_iy$ and $y \in C(u, S)$ then $x \notin C(u, S)$, for $x, y \in S$.

Arrow's impossibility theorem holds when there is more than three alternatives. However, when there are only two alternatives, Arrow has proved that it is possible to construct a social choice mechanism satisfies all the conditions if we relax condition 1 to allow only two alternatives in the universal set.

We define a majority decision method as:

M: ${}_xR_y$ if and only if $N(x, y) \geq N(y, x)$.[3]

M satisfies Axiom I: It is because either $N(x, y) \geq N(y, x)$ or $N(x, y) \geq N(y, x)$ holds. It means, either ${}_xR_y$ or ${}_yR_x$.

M satisfies Axiom II: Since there are only two alternatives, either z=x or z=y. If z=x, the Axiom becomes: ${}_xR_y$ and ${}_yR_x$ implies ${}_xR_x$. Since ${}_xR_x$ is equivalent to the proposition $N(x, x) \geq N(x, x)$. Therefore, it is obviously true. If z=y. the Axiom becomes ${}_xR_y$ and ${}_yR_y$ implies ${}_xR_y$. Again, it is obviously true.

M satisfies condition U: Only two possible ordering of alternatives: ${}_xR_y$ or ${}_yR_x$. Thus, M can yield a social ordering for any preference profile of individuals.

M satisfies condition P: If every individual prefers x to y, then $N(x, y) \geq N(y, x)$. We thus obtain ${}_xR_y$.

M satisfies condition I: If S=[x, y], then no irrelevant alternatives in the universal set. If S=[x], then the chosen set is the entire environment S. No alteration of the chosen set in s is possible.

M satisfies condition N: Suppose only a single individual prefers x to y, i.e. N(x, y)=1, and all other individuals not prefers x to y, i.e. $N(y, x) \geq 1$. We henceforth obtain: $N(x, y) \geq N(y, x)$. It means, not ${}_xP_y$.

Arrow suggests a Theorem 1: the Possibility Theorem for Two Alternatives. It says, "[i]f the total number of alternatives is two, the method of majority decision is a social welfare function which satisfies Condition 2–5 and yields a social ordering of the two alternatives for every set of individual orderings."[4] Arrow also adds. "Theorem 1 is, in a sense, the logical foundation of the Anglo-American Two-party system."[5]

Therefore, it is no need to assess electoral systems based on majority decision rule when the total number of candidates in the election is only two. The assessments of electoral systems are on the assumption that the choice environment is containing more than or equal to three candidates.

Before we start the assessment, let us first have a general survey of electoral systems currently practised in democratic countries worldwide.

CHAPTER NINE

GENERAL SURVEY OF ELECTORAL SYSTEMS

Arrow has pointed out that there are two kinds of social choice mechanism. "In a capitalist democracy there are essentially two methods by which social choices can be made: voting, typically used to make "political" decisions, and the market mechanism, typically used to make "economic" decisions."[1] Thus, electoral system is a typical social choice mechanism to make political decisions in democratic countries.

There are many types of electoral system, but they all comprise four basic elements: electoral formulas, district magnitudes, ballot structures and electoral thresholds. The first one - electoral law, is the key factor governing the process and consequences of an election. Classification of electoral systems are mainly according to this dimension. The second one is the district magnitude. It denotes the number of candidates to be elected in a district. District magnitude is neither a geographical area nor the number of voters in it, but the number of seats assigned to the district. Since fractional seats do not exist, district magnitude vary as positive integers. The third element of electoral system is the ballot structure. There are two types of it: categorical or ordinal. "Categorical ballots ask the voter to decide which one of the parties he prefers. The ballot forces him to say that he prefers one party in parliament as opposed to all others. He cannot equivocate; he cannot qualify his decision."[2] Categorical ballots are the normal ballots in the sense that they insist that the voter expresses his preference in a dichotomy. (that is, no rank-order of candidates) Ordinal ballots allow a voter expresses his preference in an order. Voters hence can make a rank-order among candidates or parties. "He may thus say that he prefers Party A most, Party C second, and so forth. The voter need not opt in favor of any single contestant."[3] The last component of electoral system is the electoral threshold. In order not to make small parties to win election, all countries have a threshold established to limit small parties. Electoral threshold is defined as the minimum number of seats won at the district level and/or a minimum percentage of the total national vote for having a seat in the parliament.

Electoral systems are the practical aggregation device used to convert thousands of votes cast by electors into limited seats in a legislature. Electoral systems can be classified by the way they allocate seats. Broadly speaking, there are four ways in which this can be done: Seats allocate to those candidates obtaining a plurality, or majority of the vote, or semi-proportionately, or proportionately. Therefore, we have plurality systems, majority systems, semi-proportional systems and proportional systems of election.

In addition, there are many methods of allocating seats proportionately. Their differences are based upon preferential voting in multi-member constituencies and party lists voting. Party list systems can be further subdivided into those which allocate seats nationally, and those allocation within multi-member constituencies. They also can further sub-subdivided according to the methods by which candidates are chosen. An electoral system may require electors to vote only for a party list or offer certain degree of choice among candidates within a party list, or even across party lists. Here, I try to lay out the various electoral systems as in the figure below:

A Classification of Electoral Systems

Figure 1.

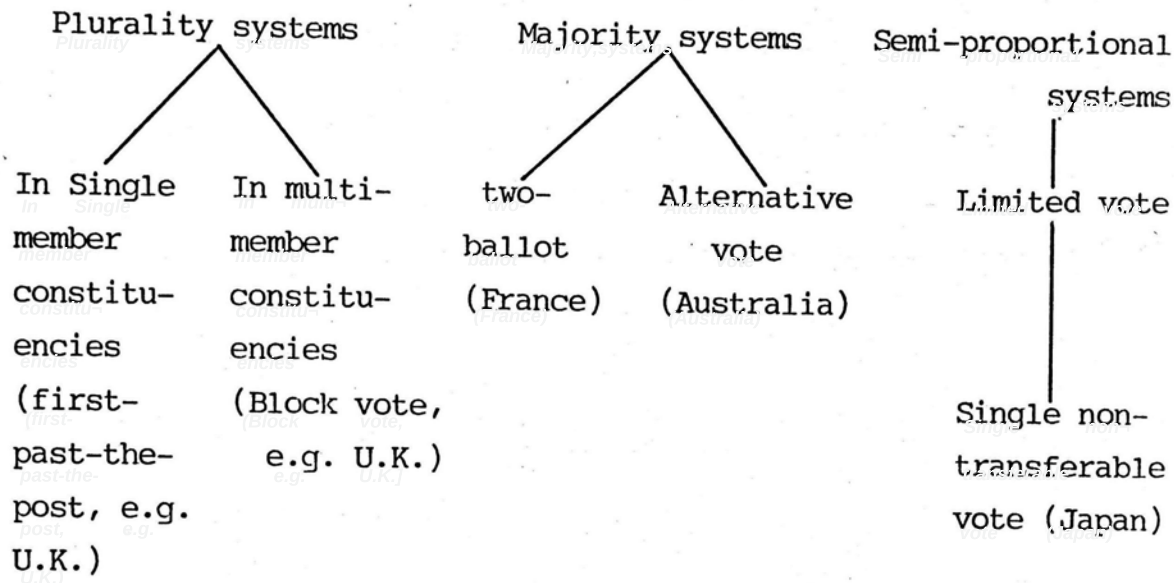


Figure 2.

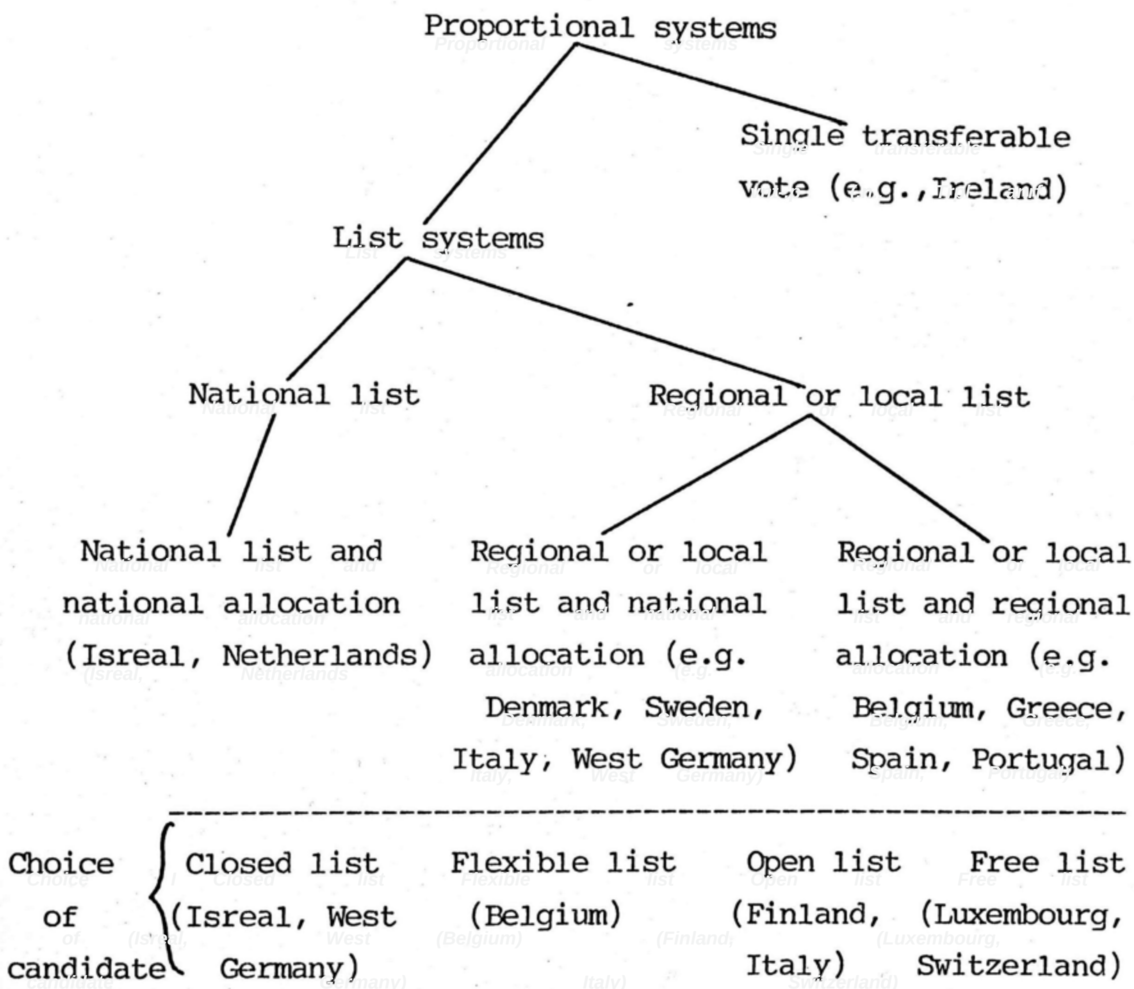


Figure 3.

A Classification of the Electoral Systems for the Election of the First Chambers in 22 Democratic Countries, 1945-1980

	Plurality System:	Canada New Zealand United Kingdom United States
Majority Systems	Two-Ballot System:	France V
	Alternative Vote:	Australia
Semi-proportional Systems	Single non-transferable Vote:	Japan
Proportion Representation Systems	LIst Systems:	Austria Belgium Denmark Finland France IV Germany Iceland Isreal Italy Luxembourg Netherlands Norway Sweden Switzerland
	Single Transferable Vote:	Ireland

[4]

SECTION III

ELECTORAL SYSTEMS

CONDITIONS FOR ELECTORAL SYSTEMS

GENERAL SURVEY OF ELECTORAL SYSTEMS

§9.1 Plurality Systems

Plurality system, as it developed in Britain and is used in countries which have come under British political influences- the Commonwealth countries. Plurality system, or it can be called the first-past-the-post system, is the oldest type of electoral system. It awards a seat to a candidate who gets a plurality of votes in a given constituency. Plurality system is a disproportional representation mechanism. It enhances a major party over-represented in the parliament. Thus it is likely to yield single-party government without the need for coalitions. In a simple plurality system, each constituency normally elected a single member. The plurality system in single-member constituencies is the simplest to operate. Each elector is offered a list of candidates on a ballot paper for the relevant constituency and is asked to select one. The candidate receives most of the votes is declared elected.

Electoral systems used in a multi-member constituency, whereby each elector has as many votes as there are candidates to be elected, is generally known as block vote. This system, is used in various local elections in Britain. This is the plurality system in multi-member constituencies, in which the n candidates receiving most votes are declared as the winners if n candidates are to be elected

The distinctive feature of plurality system is that the share of the vote needed to win a given seat cannot be known in advance. It depends on the number of candidates and the distribution of votes among candidates in a given constituency.

§9.2 Majority Systems

Majority systems seek to eliminate the possibility of a candidate elected on a minority of vote. There are two types of majority systems: the alternative vote and two-ballot system. In the alternative vote system, the winning candidate in a constituency must secure an absolute majority (i.e. at least one more vote than 50% of the total). This is a preferential system in single-member constituencies. Electors are required to rank alternative candidates in order of preferences. By doing so, an elector is asked to indicate "how he would vote if his favourite candidate were defeated and he had to choose again among the remaining candidates. This is the alternative vote."[5] Absolute majority of votes may be obtained on the first count, but if no candidate has an absolute majority of the vote, the candidate with lowest first-preference votes is eliminated and his votes will be redistributed according to the second preference recorded on those ballot papers. If this does not produce an absolute majority, then the next lowest candidate is eliminated and his votes are redistributed. This process continues until one candidate has an absolute majority of the vote. The alternative vote is a method to guarantee majority representation in a constituency; however, it remains a system of disproportional representation.

The two-ballot system, as its name implies, after the first ballot has been held, if no candidate has won an absolute majority of the vote, a second ballot is performed. The results of the first ballot determine which candidates in a constituency may participate in the second ballot. In National Assembly elections in the Fifth Republic of France, only those candidates who have gained the votes of more than 12½% of the registered electorate can participate in the second ballot. For presidential elections, only the two highest ranking candidates can contest in the second ballot. Both types of the majority system we have discussed is to elect representatives by an absolute majority of the votes and are therefore also disproportional in nature.

The fundamental features of the plurality and majority systems are that: (1) they are disproportional; (2) the number of seats which a party receives depends not only upon the number of votes which it gains, but also the distribution of votes. Electoral systems used in a single-member constituencies can not guarantee proportional representation since votes for those losing candidates are wasted in the sense that these votes bring no seats.

§9.3 Semi-Proportional Systems

If a elector has votes less than the number of candidates to be elected in a multi-constituency, then the system of election in used is known as limited vote. Limited vote is only used in Japan today. In Japan, each elector has one vote, although constituencies are multi-member. There is no mechanism for transferring votes from one candidate to another. Those declared elected are simply those n candidates receiving most of the votes in a n-member constituency. This system secure minority representation in two ways: First, since an elector only has one vote, candidates coming from the same party are to some extent have to compete against each other for the single vote form an elector. In this situation, a minority party will gain representation if it puts up only one candidate and secures just over a quarter of the vote in a three-member constituency.(If four candidates run for three seats, the last one must receive votes less than a quarter of the total, therefore, anyone secures a quarter of total votes must not be the last and hence will be allocated a seat) Second, this system is less disproportional than plurality or majority systems since it forces even the strongest party to nominate fewer candidates than there are seats to be allocated so that its supporting candidates are more likely to have evenly distribution of votes in a greater proportion of the total.

Although the single non-transferable vote facilitates the representation of minorities, they do not ensure a proportional relationship between votes and seats. This point can be illustrated from the 1969 House of Representatives election in Japan, where each constituency returning three, four or five members. (with the exception of one single-member constituency among 130 constituencies in Japan) [6]

	%Votes	Seats Won	Seats in proportion to votes	Extra Seats	%Seats
Liberal Democratic P.	47.6	288	232	+56	59.2
Socialists	21.4	90	104	-14	18.5
Democratic Socialist P.	7.7	31	37	-6	6.4
Komeito	10.9	47	53	-6	9.7
Communists	6.8	14	33	-19	2.9
Other small parties	0.1	0	0	0	0.0
Independents	5.5	16	27	-11	3.3
Totals	100.0	486	486	0	0.0

§9.4 Proportional Representation Systems

In proportional representation systems, the purpose of an election is to produce as closely as possible a match between a party's share of the votes and its share of representation in the national parliament. Therefore, the necessary condition for proportional representation systems is a multi-member constituency. There are two types of proportional systems: list systems and single transferable vote. Here I will discuss them separately in the following.

§9.4.1 Type I: List Systems

List systems, are used by every continental European democratic countries except France. It is another main type of proportional representation electoral systems. The European countries modify this system into various forms by different ways of designing list, allocation of seats and restrictions to the elector's choice among candidates. In fact, list systems are of many types. However, list system can be classified according to three criteria: (1) Whether the list is national or sub-national (i.e. regional or local); (2) whether the allocation of seats is at national level or in multi-member constituencies in regional or local level; (3) whether the system allows electors to choose between different candidates of their preferred party (i.e. flexible or open list), or even allow electors to choose across parties (i.e. free list), or whether it confines electors to vote for a party list with the order of candidates being determined by the party. I will discuss these criteria respectively bellow:

(1) National list systems are used only by Isreal and the Netherlands. Isreal goes to the extreme accuracy in party representation for the whole country is one constituency. In Isreal each party presents a list of candidates in an order decided by the party and votes are cast for the whole list. In the Netherlands, constituencies do not determine how many seats each party wins, they may determine which candidates fill the party seats. Countries using national list allocate seats proportionately at national level.

(2) Countries using regional or local constituencies but allocating seats proportionately at the national level include Germany, Denmark and Italy. National proportionality is secured through the allocation of supplementary seats from a national pool. Since a number of seats may be reserved in a national pool and allocated to the under-represented parties, this method can be used to correct the deviations from proportionality caused by small district magnitudes.

(3) List systems may or may not allow electors to choose between candidates of the same party. Isreal is the country where there is no choice at all. This type of list system is known as the closed list system. Most countries allow some choice of candidates, but this is often very limited. In Belgian, some limited choice is allowed. Electors can either vote for the list in which the order of candidates are decided by the party, or he may vote for a particular candidate by marking the preferred candidate's name on the list. This type is known as the flexible list system. In Finland, electors can have a greater degree of electoral choice. There is not an ordered list presented to electors, instead, but a series of names in alphabetical order. Electors are asked to mark a space beside the candidate they choose. This is an example of the open list system. Finally, there is a free list system in Switzerland and Luxembourg, where, again the candidates are not put in any order of preference by the parties. There, electors have not only one vote as the case in Finland, but as many votes as there are candidates to be elected. An elector may cast his votes for candidates of different parties and cumulate two votes on any one candidate if he wishes, this is a kind of approval voting in which intensity of preference may be expressed. Although both the open list and the free list systems enable an elector to rank order candidates on the party list, they still share a basic feature of list systems, "that every vote (whether or not given in the first instance to an individual candidate) is, automatically and without further reference to the voter's wishes, added to the total of the list on which that candidate appears."[7] Therefore, elector's vote for one candidate on a party list may help elect another candidate on the same list, whom the elector might not support or even disapprove of. This situation can never happen in a single transferable vote system; however, list systems are likely to yield a greater degree of proportionality than the single transferable vote, especially when allocation of seats is at national level.[8]

§9.4.2 Type II: Single Transferable Vote

The ballot structure of the single transferable vote is ordinal. It is a preferential electoral system in multi-member constituencies. It has two features: First, it secures proportional representation of political opinion. Second, it enables electors to choose candidates within, as well as between parties. Representation of opinion would be secure since it ensure that the number of votes wasted is minimised and that as many as possible electors are able to elect their representatives. Gudgin and Taylor (1974) suggest, votes can be classified into three categories according to their location:

(1) Wasted votes. These are the votes which bring no seats, because they are cast to candidates or parties which lose in constituencies.

(2) Excess votes. These are the votes also bring no seats, because they are extra to the number needed to win the constituencies in which it is successful.

(3) Effective votes. These are the votes which win seats, and are calculated as one vote more than the total won by the opposition. [9]

In a plurality or majority systems, a constituency is won by the candidate or party getting most votes. If there are two parties contesting in a single-member constituency and there are 10,000 votes from the electorate, then 5,001 votes of the total, which is the number of effective votes, will be sufficient for victory; any more votes are excess, since they do not bring extra seats for the party. On the other hand, if the party receives fewer than 5,001 votes, these votes are wasted since they do not bring representation. Therefore, the aim of the single transferable vote is to attempt to minimize the number of wasted and excess votes. In single transferable vote system, votes would be secured not to be wasted simply because they would be transferred. (The only wasted votes would be those cast for the runner-up and the voter had refused to indicate a full list of preferences. This makes those non-transferable) In this system, electors are asked to rank all candidates in order of preferences, candidates have to obtain a certain number of votes to be elected. In order to minimize wasted votes as possible, if a candidate wins more votes than needed, the excess votes are transferred to other candidates in proportion to the next lower preferences recorded on each ballot paper cast to the elected candidate. In this way, each elector would be represented by a candidate of his choice. The single transferable vote has been found to yield a high degree of proportionality, although not higher than the list systems. [10]

SECTION IV

ANALYSIS OF ELECTORAL SYSTEMS

A SUMMARY ANALYSIS OF ELECTORAL SYSTEMS

CHAPTER TEN

A SUMMARY ANALYSIS OF ELECTORAL SYSTEMS

In the present century, the most notable contribution to the theory of elections was made by Kenneth Arrow. Arrow's Impossibility Theorem shows that no existing deomcratic electoral systems can simultaneously satisfy a small set of reasonable democratic conditions. It means that electoral systems can break Rule U, C, I, P and even Rule N.

In democratic countries worldwide, a wide variety of electoral systems are being practised. In what sense are these electoral systems unreasonable? What Arrow's conditions do these systems violate? In order to find out the answer, I will examine the plurality systems, two-ballot system, alternative vote system, single non-transferable vote system, list proportional representation systems and also the alternative vote system. Although conditions other than Arrow's conditions have been proposed,[1] the analyses performed here, are mainly on the basis of Arrow's conditions.

§10.1 Plurality Electoral Systems

In a plurality system, each eligible voter has as many votes as there are seats to be filled. Usually, only a single member is elected in each constituency. Candidate x is a winner under the plurality system if x obtains more votes than every other candidates in an election. i.e. C(u, S)=[x: $x \in S$ and $t_x \gg t_y$ for all y S]. Whenever there is a tie, it can be resolved by some tie-breaking methods such as drawing lots. Plurality system is a non-ranked system. Voters need not to rank candidates on the ballot paper. Just a cross or a tick is enough for voters making a choice. It is assumed that electors cast their votes to a candidate of their first preference.

(1) Plurality system satisfies Rule U: For a non-paradox preference profile, obviously plurality system can yield a chosen set. Moreover, even in a paradox preference profile, it produces a tie. Tie can be resolved by some tie-breaking methods.

(2) Plurality system violates Rule C: Consider the preference profile: u=[R_1, R_2, R_3] where R_1=xwyz, R_2=ywzx, R_3=zxyw, and X=[x, y, z, w]. According to the definition of social choice function stipulated in Rule C, a social choice is consistent if $x \in$ C(u, [x, y]) and $y \in$ C(u, [y, z]) then $x \in$ C(u, [x, y]). However, in our case we have [z]=C(u, [z, x]), [w]=C(u, [x, w]) and [w]=C(u, [z, w]). This violates the Rule C.

(3) Plurality system satisfies Rule P: Since when all individuals prefer x to y, y can not be the one who obtains most votes and hence cannot be the winner.

(4) Plurality system satisfies Rule I: Let us consider two preference profiles, u=[xyz, yzx, zxy] and u'=[xyz, xyz, zxy]

where X=[x, y, z]. For a choice environment S=[x, y], u^S=[xy, yx, xy] and u'^S=[xy, yx, xy] Since, $R_i{}^S = R'_i{}^S$ and also C(u, [xy]) = C(u', [xy]) = [x]. Therefore, plurality system satisfies the Rule I.

(5) Plurality system satisfies Rule N: It is because no single individual's choice can become society's choice. A candidate emerges as a winner only when he is preferred by most of the voters. Therefore, plurality system garantee the Nondictatorship condition.

Plurality system can produce an extremely weak representation when there are many candidates. Consider the case with four candidates and 40,000 voters:

Voter group A (11,000)	$xPyPzPw$
Voter group B (10,000)	$yPzPwPx$
Voter group C (10,000)	$zPyPwPx$
Voter group D (9,000)	$wPyPzPx$

In plurality system, candidate x will be elected with 11,000 first-preference votes, candidates y, z and w will be rejected with 10,000, 10,000 and 9,000 votes respectively. The elected candidate does not have majority support. In fact, only 11, 000 out of total 40,000 votes prefers candidate x and is then declared to be elected even though the majority, 29,000 votes ranks x as the last in their preference. Moreover, the total wasted votes is up to 3/4 of the total votes.

The wasted votes phenomenon indeed have psychological effect on voter's behaviour. A voter, whose favorite seems no chance of winning, may vote strategically for another candidate he approves of who has a better chance. For example, the popular-vote percentages in the 1980 presidential election in the

United States were 51 for Ronald Regan, 41 for Jimmy Carter, and 7 for John Anderson. Brams and Fishburn (1982), using available data from several sources, have estimated that if the same voters had voted their true preferences, then the percentages would have been about 40 for Regan, 35 for Carter, and 24 for Anderson. More than 70% of those who favored Anderson voted for either Reagan or Carter. Thus the wasted votes phenomenon induced voters to vote for two promising candidates and turned the election into a two-candidate race between Reagan and Carter. [2]

In addition, since the number of votes required to win cannot be known in advance, the plurality system has been frequently blamed for not reflecting in the elected candidates the various opinions of the electorate.

§10.2 Two-Ballot System

By using only first place preferences, plurality system does not take into account the information about the preferences of voters. It may select a candidate who is ranked the last by the majority. In order to overcome this deficiency, a double ballot system is introduced. In the two-ballot system, which is normally restricted to single-member constituencies, a second round of election is held if no candidate gains more than 50 per cent of the votes cast in the first ballot. In the second ballot, normally only two leading candidates are admitted. It has been the system used most frequently in France in parliamentary elections until 1986. In presidential elections, two-ballot system is directly used to elect their presidents in Australia, Cyprus, Finland, Ireland and France.[3]

A majority winner is that the winning candidate or "the winning party has defeated the entire field of opposition; no combination of opponents can match its numerical strength. If Party A has a majority, then:

$$t_a > (t_b + t_c + \ldots t_n)\text{"}; \quad \text{where } t_a + t_b + \ldots t_n = V \quad [4]$$

Thus, the chosen set for the first ballot can be defined as:

$$C(u, S) = [a: a \in S \text{ and } t_a > (t_b + t_c + \ldots t_n)]$$

For the second ballot, plurality electoral rule is in use. If there are exactly two candidates, the plurality voting rule makes the same choice as simple majority rule.

(1) Two-ballot system satisfies Rule U: It responses to all logically combination of preference orderings of individuals. Tie is resolved by some tie-breaking methods.

(2) Two-ballot system violates Rule C: Two-ballot system is inconsistent as can be seen from the following case, where X=[x, y, z], u=[R_1, R_2, ... R_{11}], in which R_1=zyx, R_2 and R_3=xyz, R_4 to R_6=yzx, R_7=yxz, R_8 and R_9=zxy, R_{10} and R_{11}=xyz. The effective votes for being a majority winner are 6. We can see that C(u, [x, y])=[x], C(u, [y, z])=[y] but C(u, [x, z])=[z]. This violates Rule C in the first ballot.

In this two stages choice procedure, assuming only two leading candidates can remain in the second ballot. Also, voters will vote according to their ordering of candidates. Since in the first ballot no majority winner emerges, candidate z will be eliminated. In the second ballot, candidate y will win x by 6 to 5 votes. i.e. C(u, S)=[y]. However, if S'=[x, z], the chosen set becomes: C(u, S')=[z]. Since C(u, S) $\neq$ C(u, S'). It also violates Rule C.

(3) Two-ballot system satisfies Rule P: The reason is the same as the plurality system. Since when all individuals prefer x to y, y cannot be the one who obtains majority of votes.

(4) Two-ballot system satisfies Rule I: Two-ballot system only considers candidates within the choice environment, therefore no irrelevant alternatives can affect the choice outcomes. (Rule I will be violated only when the voting procedure will take the alternatives outside the choice environment into consideration.)

(5) Two-ballot system satisfies Rule N: It is because two-ballot system only yields a unique winner, therefore no candidates can be elected only by a single individual's preference.

Two-ballot system fails the same test as the plurality system. However, it is better than plurality system because it guarantees the winner is the one who will not be ranked last by most of the voters.

§10.3 Alternative Vote System

The alternative vote system is a true majority system. In which voters are asked to indicate their first preference, second preference, and so on among candidates. Voter's vote will be transferred to his next preferred candidate if his favourite were defeated. This system on the one hand solves the wasted votes problem in the plurality system, on the other hand it secures an absolute majority winner in a single ballot. It is also more easier to implement than the two-ballot system.

The alternative vote system is currently used in Australia. "The present situation [1986] is that the lower houses of New South Wales, Victoria, Queensland, South Australia, and Western Australia; the federal House of Representatives; and the Legislative Assembly of the Northern Territory are elected from single-member districts by majority-preferential methods. [alternative vote methods] The upper houses (Legislative Councils) of Victoria and Western Australia are elected from two-member provinces with overlapping-term provisions so that in effect, a single-member-district system applies. The Legislative Council of Tasmania is elected from single-member districts by a majority-preferential method. The House of Assembly of Tasmania, the Legislative Councils of New South Wales and South Australia, and the federal Senate are elected by quota-preferential methods. [single transferable methods] Thus, 10 of the 14 parliamentary bodies are elected by majority-preferential methods and 4 by quota-preferential methods."[5]

The operation of this system can be illustrated in the following example:

Assumme we have: $X=[x, y, z, w]$, $u=[R_1, R_2, \ldots R_{21}]$.

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
x	y	z	y	x	w
y	x	y	w	w	x
w	w	w	z	z	z
z	z	x	x	y	y

Since no candidate obtains a majority of first place votes, we eliminate w: the candidate with the lowest number of first preferences. The preference orderings become:

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
x	y	z	y	x	x
y	x	y	z	z	z
z	z	x	x	y	y

There is still no majority winner, and elimination procedure is repeated by excluding candidate z. Now the preference orderings become:

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
x	y	y	y	x	x
y	x	x	x	y	y

Candidate y beats x by 11 to 10 votes and hence is the majority winner in the alternative vote system.

(1) Alternative vote system satisfies Rule U: It responses to all logically combination of preference orderings of individuals.

(2) Alternative vote system violates Rule C: Alternative vote system is a multi-stage voting procedure. From the above example, we obtain C(u, S)=[y], where S=[x, y, z, w]. However, when S'=[y, z], the preference orderings become:

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
y	y	z	y	z	z
z	z	y	z	y	y

The chosen set: C(u, S') is [z] since z beats y by 11 to 10 votes. For $y \in S' \subset S$ and C(u, S)=[y], but C(u, S')=[z]. Thus, we may conclude that alternative vote system is inconsistent.

(3) Alternative vote system satisfies Rule P: Alternative vote method is based on the number of first ranks given to various candidates by voters and also its chosen set is unique. It guarantees that if everyone unanimously prefers one candidate, say x, to another, say y, then y would not be the winner.

(4) Alternative vote system satisfies Rule I: Alternative vote system only considers candidates inside the choice environment, thus no irrelevant alternatives have to worry about.

(5) Alternative vote system satisfies Rule N: Alternative vote system always yields a majority winner, therefore, no single individual's favourite can emerge in the chosen set.

The problem of alternative vote system, just the same as all other plurality and majority systems, is that it yields disproportionate electoral outcomes.[6]

§10.4 Single Non-transferable Vote System

This system is currently practised in Japan in national-level elections of the House of Representatives. Most of the districts in Japan are three-member, four-member and five member districts. At present, there are 130 constituencies throughout Japan, all except one are multi-member constituencies. Of the 511 representatives, 510 are elected by single non-transferable vote system (SNTV); one is elected in a single-member constituency, whch is in fact a plurality instead of a SNTV.[7]

SNTV is a special case of the limited vote in which each voter has only one vote in a multi-member constituency. SNTV is different from plurality and majority systems in two aspects. First, the plurality and majority systems normally are applied in single-member constituencies whereas the SNTV requires multi-member constituencies. Second, SNTV facilitates minority representation. This effect, however, cannot be achieved by either plurality or majority systems.

The SNTV achieves minority representation can be explained in terms of its electoral threshold. According to Rae, Hanby, and Loosemore (1971), there are two types of threshold: the threshold of representation and the threshold of exclusion. The threshold of exclusion is the percentage of the vote that will guarantee the winning of a seat even under the most unfavorable situation. For example, the most unfavorable situation for a minority party to face in a four-member constituency is that one large majority party with four candidates who each receive exactly equal shares of the total votes in relative high proportions. however, if the this minority party has only one candidate who receives slightly more than twenty percent of the votes, the candidate from this minority party is assured of a seat in the election. It is possible that a minority party wins a seat with fewer votes than the threshold of exclusion. The threshold of representation is the vote percentage that may be sufficient for a party to win a seat under the most favorable conditions. For instance, if the majority of votes gained by a large party concentrates to a single major candidate, the other two candidates can be elected with less votes than the threshold of exclusion. Minority parties thus can gain seats when conditions are favorable to them.

Single non-transferable vote system, in fact, is the application of the plurality method in single-member constituencies to multi-member constituencies. The ballot structure for both SNTV and the first-past-the-post systems are the same. However, the chosen set in SNTV is not a single winner, instead, the size of the chosen set depends on the magnitude of an electoral district. Normally, it ranges from three to five.

SNTV system can be defined as:

$C(u, S)=[x_1, x_2, \ldots, x_i: (x_1, x_2, \ldots, x_i) \in S$ and $t_{x_i} > t_{x_{i+1}}$ for district magnitude $= i]$;
where, $t_{x_1} > t_{x_2} \ldots > t_{x_i} \ldots > t_{x_n}$; and $t_{x_1} + \ldots + t_{x_i} + \ldots + t_{x_n} = V$

(1) SNTV system satisfies Rule U: It is because SNTV responses to all logically combination of preference orderings of voters.

(2) SNTV system violates Rule C: the reason for this is just the same as the plurality system. It can be seen from the following case, where $X=[x, y, z]$, $u=[R_1, R_2, R_3]$, in which R_1=xyzw, R_2=ywzx, R_3=zxyw. We can show that for a three-member constituency, C(u, [z, x])=[z], C(u, [x, w])=[w]; But we find that C(u, [z, w])=[w] . This violates Rule C

(3) SNTV system satisfies Rule P: It is obvious since when all individuals prefer x to y, y gains no vote and therefore will not be elected.

(4) SNTV system satisfies Rule I: SNTV system only considers candidates within the choice environment. Therefore, no irrelevant alternatives can affect the choice outcomes.

(5) SNTV system violates Rule N: Suppose in a two-member constituency, where X=[x, y, z, w], u=[R_1, R_2, R_3, R_4], in which R_1=xyzw, R_2=xzyw, R_3=xwyz and R_4=wxzy. We obtain C(u, S)=[x], and also all individuals prefer x to w except individual 4. However, w$\in$C(u, S). Thus, individual 4 is a dictator.

Although SNTV can yield a higher proportional result, it cannot solve the wasted votes problems. The representation of an elected candidate can be quite low. Moreover, it does not seem to have much improvement for plurality and majority systems.

§10.5 List Proportional Representation Systems

The Principle of this type of electoral systems are to allocate seats to any party in the legislature as closely as possible to the share of the votes which it has won. Ideally, the share of the total seats should be in direct proportional to the share of the total votes.

The principle of proportionate share can be defined as:

S = t.[8]

For example, in a constituency with 42 million voters electing a legislature of 100 members, the ideal distribution of seats corresponding to votes for parties will be:

Pure proportional representation

	Millions of voters	% share of vote	No. of seats
Party x:	12	28.6	29
z:	11	26.2	26
y:	10	23.8	24
w:	9	21.4	21
	42	100.0	100 [9]

Since seats are distributed proportional to the distribution of votes, so this requires more than one vacancy, and multimember constituencies are necessary.

The list systems are used in majority of democractic countries.(eg. Denmark, Finland, Germany, Iceland, Isreal, Italy) There are some variations in list systems, but they are basically the same that the parties nominate lists of candidates in multimember constituencies, voters cast their ballots for one party list or the other. Seats are allocated to the party lists in proportion to the share of votes that party have obtained.

List systems may be subdivided further according to the mathematical formula used to translate votes into seats. The subcategories are the Largest Remainder and the Highest Average Methods.

§10.5.1 The Largest Remainder List System

The largest remainder method is the simplest means of allocation. It begins with the computation of a quota, by which a party must achieve for granting a seat. The most commonly used quota is the Hare quota, which is defined as:

$$\text{Hare quota } (q) = \frac{V}{M} \qquad [10]$$

(q) is equal to the number of votes cast divided by the number of seats assigned to the district.

In the case of a four-member constituency where 20,000 votes have been cast, the Hare quota will be 20,000/4, which is equal to 5,000 votes.

Case 1: Four-member Constituency, 20,000 votes cast.[11]
Hare quota: 5,000 votes

Party	Votes	Quota	Seats	Remainder	Seats	Total Seats
A	8,200	5,000	1	32,00	1	2
B	6,100	5,000	1	1,100	0	1
C	3,000	-	0	3,000	1	1
D	2,700	-	0	2,700	0	0
Total	20,000		2		2	4

In the above case, only two of the four parties achieve the quota. Therefore, only two of the four seats can be directly allocated; one to party A and the other to party B. Under the largest remainder system, the third seat goes to party A and the fourth seat to party C. Hence, the largest remainder system is favourable to smaller parties. It should be noticed that party C gets as many seats as party B while getting less than half of B's total votes.

§10.5.2 The Highest Average List System:

The highest average system is commonly known as the D'Hont rule, named after its inventor. Its central idea is to divide each party's votes by successive divisors, the D'Hondt divisors: 1, 2, 3, 4, 5, etc., and then allocate the seats to the parties in descending order of quotients. The purpose is to secure that when all the seats have been allocated, the average number of votes required to win one seat is as nearly as possible the same for each party. Let us use the same electoral results in the previous case to illustrate this system:

Case 2: Four-member Constituency, 20,000 votes cast.[12]
Division by D'Hondt divisors

Party	Votes	Divisor:1	Divisor:2	Divisor:3	Total Seats
A	8,200	8,200(1)	4,100(3)	2,733	2
B	6,100	6,100(2)	3,050(4)	2,033	2
C	3,000	3,000	1,500	1,000	0
D	2,700	2,700	1,350	900	0
Total	20,000				4

We can see that under the D'Hondt system the allocation of seats is different from that of Largest remainder system. The first seat is allocated to party A, the second to party B, the third to party A and the fourth to party B. Party C and D are left without seat. D'hondt system is unfavorable to smaller parties. For a smaller party, the remainder for a seat will be a higher proportion of its total votes, and therefore will be more costly to it.

In view of this unfairness, an alternative divisors, the Sainte-Lague divisors has been adopted in several Scandinavian countries. The Sainte-Lague divisors involves dividing each party's votes by 1.4, 3, 5, 7, etc., instead of by 1, 2, 3, 4, etc. The Sainte-Lague divisors differs from the D'Hondt method in two ways: (1) the first divisor is larger by 40%, and (2) the relative distances between the number of the series are larger.

In the case of the distance between the second and the third divisors, the difference between Lague and D'Hondt are:

Lague: 3 to 5. (5-3=2) relative distance=2/5 = 0.40
D'Hondt: 2 to 3. (3-2=1) relative distance=1/3 = 0.33 [13]

The Sainte-Lague method is in favor of medium-sized parties by lowering the advantage obtained under D'Hondt procedures by the largest party and by raising the threshold at which small parties begin to win seats. (notice that the first divisor is 1.4 instead of 1.)

The largest remainder and the highest average system are not mutually exclusive. It is possible to combine both methods by first using Hare quota for direct allocation of seats and than allocating the remaining seats by D'Hondt method or Saint-Lague method. The selection of electoral formula is based on whether greater or lesser advantage should be given to large, small or medium-sized parties.

In party list systems, only voters' first preferences are counted, the choice for voters, therefore must refer to political parties, or precisely to the candidate list of political parties. Although there are many ways of allocating seats to candidates within a party (eg. closed list, flexible list, open list), voters are fundamentally voting for a party list.

For our convenience, let us consider voters normally vote for parties, not candidates. If we examine the preference profile of individuals over parties, we will find that an intransitive social ordering by pairwise comparison of parties may occur.

Recalling the previous example for a pure proportional representation, we try to take individuals' preference pattern into consideration this time:

Pure proportional representation

Group	Voting	Millions of voters	% share of vote	No. of seats
A	$_xP_wP_zP_y$	12	28.6	29
B	$_zP_wP_yP_x$	11	26.2	26
C	$_yP_zP_xP_w$	10	23.8	24
D	$_wP_yP_xP_z$	9	21.4	21
		42	100.0	100

(1) Party list systems violate Rule C: From the above preference profile, we obtain C(u, [x, y])=[x], C(u, [y, z])=[y, z]. By social transitivity, we yield C(u, [x, z])=[x]. However, we also obtain C(u, [x, z])=[z]. This violates the Rule C.

The incositency can also be explained in terms of social ordering. We have (12+11+9) votes for $_xP_y$--(1), equal votes (12+10) for y and z, i.e. $_yI_z$--(2). From (1) and (2), we get $_xP_z$. But (11+10+9) votes prefer $_zP_x$. We thus obtain both the social orderings $_xP_z$ and $_zP_x$. An intransitive social ordering is resulted.

(2) Party list systems satisfy Rule U: Party list systems response to all combination of individual's preferences; even in a paradox preference profile it produces a winner.(as in the case illustrated above, it produces a winner: C(u, [x, y, z])=[x])

(3) Party list systems satisfy Rule I: Since only alternatives within the choice environment are taken into consideration, no irrelevant alternatives outside the choice environment can affect the election outcomes.

(4) Party list systems satisfy Rule P: Unanimity here would mean one party winning all the seats. This case is guaranteed in party list systems for they respond accurately to the proportion of votes received by a party.

(5) Party list systems satisfy Rule N: Normally, in party list systems a threshold is imposed in order to prevent small parties too easy to win a seat with relatively low percentage of total votes. This measure effectively prohibits any single voter becomes a 'dictator'.

§10.6 Single Transferable Vote System

Party list systems stand up well in responding to the diversity of voters opinion for they distribute seats available in a multimember-constituency among the parties in approximate proportion to the votes cast for the party lists. However, candidate's chance of being elected depends overwhelmingly on his/her party's strength in that constituency and his/her position in the party list rather on voter's opinion or knowledge about him/her. In party list systems, voters are asked to vote for a party list, that all of the candidates of the party he/she chooses are presumed to be better than all candidates in other parties. Voters' support to a party may help to elect some other candidates that he does not like. Single transferable vote system, allows voters a much larger choice of individual candidates among parties, is considered as a remedy for the disadvantages of list systems. [14]

STVS is currently practised in the Republic of Ireland. Since the constituencies are normally relatively small (3-5 members), it yields a less proportional results than most party list systems.

The STVS is an extension of the alternative vote system. STVS is designed to select more than one candidate from each constituency, while in alternative vote system only a single winner emerges. In this sense, the example used to demostrate the inconsistency of the alternative vote system can also be used to show the inconsistency of the STVS.

Before we analysis STVS in terms of Arrow's condition, we will give a brief description of how the STVS operates:

1. Voters are required to cast their votes in terms of their preference order among the candidates: first choice, second choice, third choice, etc.

2. The ballot papers are sorted and counted according to first preferences. Candidate will win a seat if he or she received enough first choice votes to pass a prescribed quota. This quota, known as the Droop Quota, can be stated as:

$$\text{Droop Quota (Q)} = \frac{\text{Total votes (V)}}{\text{Total seats (M)}} + 1 \qquad [15]$$

3. If no candidate received enough first place votes, or if some seats remained to be filled, then second place choices are counted and transferred to those candidates who need them to attain the quota. This transfer is conducted from the top--from candidates who passed the quota with more first place votes than needed. If such surplus votes do not exist, the transfer then takes place at the bottom--from candidates who received the least first place votes.

In the former case, the vote awarded to another candidate is equal to:

$$\frac{\text{elected candidate "surplus"}}{\text{total vote for elected candidate}} \times \left(\begin{array}{c}\text{second preferences} \\ \text{for the unelected} \\ \text{candidate}\end{array}\right) \qquad [16]$$

These votes are awarded at the discount way outlined in the formula above to the remaining candidates.

In the latter case, second place preferences are transferred with full weight.

The following example, which demonstrates the inconsistency of the alternative vote system, also shows that STVS is inconsistent.

Assumme we have: $S=[x, y, z, w]$, $u=[R_1, R_2, \dots R_{21}]$.

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
x	y	z	y	x	w
y	x	y	w	w	x
w	w	w	z	z	z
z	z	x	x	y	y

The 21 voters in the district distributed and rank their preferences among the candidates as shown in the table above.

In order to computed a winner, the first step is to find the Droop Quota: total votes (V) =21

total seats (M) = 2

$$Q = \frac{21}{2+1} + 1 = 8$$

Any candidate with 8 votes is elected.

1st count: first preferences:

x = 9
y = 6
z = 5
w = 1
21

Candidate x is elected. Transfer x's 1 surplus vote. (y gets 4/9 vote, w gets 5/9 vote.)

2nd count: first and x's second preferences:

(x = 8)
y = 6 + 4/9
z = 5
w = 1 + 5/9
21

No candidate is elected. Elimination of w and transfer of his votes. (1 votes goes to z. The 5/9 vote, is originally awarded from x, which cannot be transferred further, it is regarded as the exhausted vote.)

3rd count: first preferences, x's second preferences and w's third preferences.

(x = 8)
y = 6 + 4/9
z = 6
Exhausted = 5/9
21

z is eliminated. y is elected.

Result: x and z is elected. i.e. C(u, S)=[x, y]

(1) STVS violates Rule C: When S'=[y, z], voters preference orderings become

4 Voters	3 Voters	5 Voters	3 Voters	5 Voters	1 Voters
y	y	z	y	z	z
z	z	y	z	y	y

Therefore, C(u, [x, y])=[z].

Since $y \in C(u, S)$ but $y \notin C(u, [y, z])$, we thus conclude that the STVS is inconsistent.

(2) STVS satisfies Rule U: Just the same as the alternative vote system, it responses to all logically combination of preference orderings of individuals.

(3) STVS satisfies Rule P: STVS is based on the ranking given to various candidates by voters. If all individuals prefer x to y and x is available, the STVS certainly ranks x higher than y. Among x and y, x is selected instead of y.

(4) STVS satisfies Rule I: STVS only considers candidates inside the choice environment, therefore, it satisfies Rule I.

(5) STVS satisfies Rule N: STVS always yields a majority winner, therefore, no single individual's favourite can emerge in the chosen set.

The STVS may have many merits not found in other systems, however, due to its inconsistency, it is possible that "[a] candidate can win in each district separately, yet lose the general election in the combined districts."[17]

CONCLUSION

Arrow's Impossibility Theorem is a demonstration of the inconsistency among his conditions. The proof of the theorem employs a particular preference profile which produces a cyclical majority to show that condition I to IV lead to a contradiction of the nondictatorship condition. In part one of section I, I have demonstrated that if an individual is decisive for any pair of alternatives, then the individual is a dictator over any number of alternatives in the same set. In part two of section I, I have shown that the dictator, in fact, exists. In section II, the result obtained in section I is generalized and applied to any finite number of alternatives and any number of individuals. Thus, Arrow's theorem holds for any number of individuals and alternatives greater than or equal to three. It should be noticed that Arrow's paradox occurs only for a particular pattern of preferences profiles, not for all preference profiles. Moreover, Arrow's Impossibility Theorem is a logical result, not the result of empirical observation. Arrow examined, not particular systems, but a fully general, abstract system. Therefore, we need not to examine various systems of social choice, such as majority rule, proportional representation systems, we will know for certain that all systems having the same impossibility result as the abstract one.

The proof of Arrow's theorem is valid, therefore the assessment of Arrow's theorem focuses on various conditions that Arrow imposes on the social welfare function.

From the analysis of Arrow's conditions, we found that they are not the requirement of rationality, actually they represent a particular form of democratic believes--the populistic democratic theory.

In the discussion of the probability of Arrow's paradox, we can see that the probability varied according to different assumptions. In an impartial culture, under the assumption of equiprobability, the probability of paradox is relatively low. However, if we consider certain probability vector of preference profiles for a partial culture, the probability of paradox can be much higher. Nevertheless, with a suitable restriction on a preference profiles, the paradox would disappear.

Arrow's paradox can be avoided by restriction on the Condition of Unrestricted Domain. The restrictions which we have discussed were Black's single-peakedness and Sen's value restrictedness. However, such restrictions can avoid Arrow's paradox but do not solve it. We cannot infringe Arrow's conditions merely to avoid paradox. One of the attempt to justify the infringement of the Unrestricted Domain is proposed by Mackay. He argued that the restriction on the admission of preference profiles is analogues to the imposition of a starting point to block an infinite regress. Mackay's approach, as I have pointed out, is ethically problematic. It may not be an acceptable way of handling Arrow's problem.

Arrow has argued that aggregation of individual preferences must only base on ordering of preferences. The intensity of preference should not be allowed. The reasons are that the cardinal utility scale violates the condition of independence of irrelevant alternatives and also is meaningless. However, We 'discover' that Arrow is mistaken to think that cardinal utility scale violates the condition of independence of irrelevant alternatives. We also show that the notion of 'lottery' in constructing a cardinal-utility indicator is possible. Moreover, money can also be used as a basis for interpersonal comparison. The Preference Revealing Process is a true preference inducing method which can be employed to prevent strategic manipulation in cardinal utility scale. Thus, cardinal utility scale is acceptable in certain situations. However, it cannot solve the problem of Arrow's paradox completely. Although the cardinal utility scale can reduce the occurance of the paradox, it cannot eliminate it.

The Pareto Principle is a reiteration of the condition of positive association of individual and social values and the condition of citizens' sovereignty. Pareto Principle is implied from the notion of 'social choice'. If a choice is unanimously preferred by everyone in the society without exception, the choice must be a social choice. Therefore, the Pareto Principle is a necessary condition for social choice mechanisms. Gordon Tullock and James Buchanan suggested a unanimity rule as a necessary and sufficient condition for a social choice mechanism. However, our analysis show that both theoretical and practical problems are associated with the unanimity rule.

Strasnick has suggested that the violation of Arrow's Nondictatorship Principle may be morally justified. Strasnick argued that individuals should not be treated equally in determining a social choice, the preference of the least advantaged individuals should be given priority. The major problem of Strasnick's theory is that the choice of a single individual is not a social choice at all.

We have shown that the Axiom of transitivity can be derived from the Axiom of connectivity. It means, axiom II is not an axiom but a theorem only. The requirement of transitivity is central to Arrow's conception of rationality. However, transitivity may not be a necessary condition for rationality at both individual and group levels. At the individual level, when choice involves probabilistic thinking, violation of the transitivity should not be considered as irrational. At the group level, the violation of the social transitivity can be justified by populistic democractic theory. In a paradoxical situation, we can simply declare all alternatives are indifferent and proceed a successive voting, or we can solve the paradox by some randomizing methods.

Arrow's conditions, are a set of rules for a social welfare function, which is different from a social choice function. Hence Arrow's conditions have to reformulated so that they can be applied to a social choice function. Electoral systems are a kind of social choice function. The analysis of electoral systems are based on the reformulated version of Arrow's conditions. It has found that among the six major types of electoral systems: the plurality systems, two-ballot system, alternative vote system, single non-transferable vote system, list systems and the single transferable vote system. All of them violate Rule C. The single non-transferable vote system even violates Rule N. Since we have already argued that the infringement of the Nondictatorship Condition is unjustified, therefore, the single non-transferable vote system is not a reasonable social choice mechanism. Although the violation of Rule C can be justified. However some measures must be taken whenever there is a paradoxical or inconsistent preference profiles. A common method to overcome Arrow's problem is to vote repeatedly until a transitive social outcomes are obtained. Another way to settle the paradox is to use a randomizing method. In the categorical type of ballot systems, since voters only vote for their first choice, it is difficult to identify a paradox. However, whenever there is a paradox, it will appear as a tie votes. (It is because every candidates receive exactly equal votes of first preference from voters.) Therefore, tie-breaking method is used to solve a paradox.

We can see that all electoral systems except single non-transferable vote violate only Rule C and satisfy the rest of

Arrow's conditions. Their electoral consequences are diverse. Among these various electoral systems, the selection of them cannot merely base on Arrow's conditions. It should remember that Arrow's conditions are the minimum conditions for a rational aggregation device. Therefore, additional criteria should be used in the selection of an electoral system, such as the degree of proportionality, the acceptability among the political parties etc. However, the investigation of these criteria are beyond the scope of this thesis.

APPENDIX :

The Preferential Elimination System of Voting

This system is recently adopted in the 1988 Legislative Council elections in Hong Kong for both the functional constituencies and the electoral college. "As explained in the Green Paper, this system has advantages over the 'preferential addition' system used in the 1985 functional constituency elections, because it gives greater priority to higher preferences than to lower ones and does not carry the risk that the candidate with the fewest first preference votes might nevertheless win the election. It also has an advantage over the repeated ballot system used in the 1985 electoral college elections in that it involves only one round of voting."[1]

The electoral system proposed in the White Paper is basically the same type of the alternative vote system. The operation of this system is illustrated by examples given in the appendix C of the White Paper. This system is to ensure the elected one must be an absolute majority winner through the process of deletion of a defeated candidate and transfers of his or her votes to the remaining candidates.

Suppose an election is held in a single-constituency, there are 500 electors and with four candidates: A, B, C, D. In the first count of voting, first preference votes among candidates are as follows: [2]

A: 90 B: 180 C: 120 D: 110 Total: 500

Since no candidate obtains a majority of first place votes, candidate A, the one with least first place votes, is eliminated.

Votes cast to A are transferred according to the second preference recorded on these ballot papers. Suppose the distribution of votes are:

B: 39 C: 28 D: 19 Nil: 4 (exhausted)[3] Total: 90

Exhausted votes will not be counted in the total number of valid votes. Those votes have second preference recorded will be transferred. Therefore, the results of the second round of vote counting will be:

B: 219(180+39) C: 148 (120+28) D: 129 (110+19) Total: 496

Again no majority winner emerges, the elimination process continues, at this stage, candidate D is eliminated.

D's votes will be transferred according to their second preferences:

B:59 C:18 A:30 (exhausted) Nil:3 (exhausted) Total: 110

30 votes for candidate A is set aside as an exhausted votes, since candidate A has been eliminated. (This is different from that of alternative vote, in which the third preferences of those votes will be counted and be transferred.)

19 votes originally transferred to D from A are redistributed according to the third preferences recorded on them. Suppose the third preferences recorded on these ballot papers are:

B: 10 C: 9 Total: 19

The result of the third round of vote counting is:

B: 288 (219+59+10) C: 175 (148+18+9) Total: 463

The above election result can be put in a table as shown below:

Candidate	Count 1st	2nd	3rd
A	90	Eliminated	
B	180	180+39=219	219+59+10=288 (Elected)
C	120	120+28=148	148+18+9 =175
D	110	110+19=129	Eliminated
Exhausted Votes	0	4	3+30
Valid Votes	500	496	463
Quota (Q)	251	248	232

We should notice that the difference between the alternative vote system and this preferential elimination system is that in former system, all transferable votes will be counted, but in preferential elimination system those votes cast to an eliminated candidate will not be counted even though these votes are transferable.

The defects of this system is: suppose A is eliminated first and then D, votes recorded preferences as <u>D</u>, <u>A</u>, C, B will be eliminated while votes recorded preferences as <u>A</u>, <u>D</u>, C, B will be counted. Both D and A are eliminated, but the order of them will affect the validity of the votes. This is unfair to both candidate C and electors who votes with the preference D, A, C, B.

Suppose that the votes recorded preferences as <u>D</u>, <u>A</u>, C, B and <u>D</u>, <u>A</u>, B, C are transferable, the election result will be altered when more than 22 votes out of 30 are recorded candidate C as the third preference. Moreover, this measure may lead to a drastic decrease of percentage of valid votes to total votes. Consider the following case:

	Candidate				Total valid votes
	A	B	C	D	
1st preference votes	110	120	130	140	500

Since no majority winner emerges, A is eliminated. Suppose that the second preferences shown on these 110 votes are as follows:

Candidate of second preference	Number of Votes
B	35
C	35
D	30

The result of second round of counting becomes:

	Candidate				Total valid votes
	A	B	C	D	
1st preference votes	(110) Eliminated	120	130	140	500
2nd preference votes transferred from A		35	35	30	
Total votes obtained		155	165	170	

Again no candidate obtains an absolute majority, candidate B is eliminated. Suppose that all the 120 votes cast to candidate B are recorded A as their second preferences. According to the rules of the preferential elimination system, these votes will be set aside as an exhausted vote. It means, 120 out of the total 500 votes is eliminated. The eliminated votes consist of 24% of the total.

If two candidates obtain equal votes, the preferential elimination system suggests that the number of votes cast A as first and B as second preference is added to B. Similarly, votes cast B as first and A as second preference is added to A. The two sum are compared and the one obtains less votes is eliminated.

However, this tie-breaking mechanism does not always work. When the two sums thus obtained are the same, there is no further method mentioned in this system are available to solve the tie vote. It means, this system cannot respond to certain preference profiles and therefore violates Rule U.

Since the preferential elimination system is basically the same as the alternative vote system, therefore, the results obtained from the alternative vote system can also be applied here. It is concluded that the preferential elimination system violates Rule U and C, while satisfies Rule P, I and N.

NOTES:

NOTES FOR INTRODUCTION AND CHAPTER 1

1. See A. F. Mackay, Arrow's Theorem : the paradox of social choice, Yale University Press, 1980; p.13.
2. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.59.
3. ibid., p.23.
4. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.19.
5. ibid., 13.
6. ibid., p.25.
7. ibid., p.13.
8. ibid., p.13.
9. ibid., p.13.
10. ibid., p.20.
11. ibid., p.96.
12. ibid., pp.96,97.
13. ibid., p.25.
14. ibid., p.25.
15. ibid., p.27.
16. See A. F. Mackay, Arrow's Theorem : the paradox of social choice, Yale University Press, 1980; p.9.
17. Example from Riker and Ordeshook's An Introduction to Positive Political Theorey (1973), pp.88,89.
18. ibid., pp.89,90.
19. K. J. Arrow's ibid., p.28.
20. ibid., p.28.
21. ibid., p.29.
22. ibid., p.29.
23. ibid., p.31.
24. ibid., p.31.
25. ibid., p.31.

26. ibid., p.59.
27. ibid., p.96. The definition of decisive set given by Arrow is slightly different between his original and revised proof. Here I adopt the original definition.
28. ibid., p.96.

NOTES FOR CHAPTER 2

1. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.98.
2. ibid., p.98.

NOTES FOR CHAPTER 3

1. For detail discussion, see Wesley C. Salmon, The Foundations of Scientific Inference, University of Pittsburgh Press, 1966; pp.84, 87.
2. See Bruce. Bowen's article, "Toward an Estimate of the Frequency of Occurrence of the Paradox of Voting in U.S. Senate Roll Call Votes." in R. Niemi and H. Weisberg, eds., Probability Models of Collective Decision-making, Columbus, Ohio: Merrill, 1972.
3. See Richard G. Niemi and Herbert F. Weisberg, "A Mathematical Solution for the Probability of the Paradox of Voting", Behavioral Science, 13 (1968); pp.317, 323. Similar calculations can be found in Frank Demeyer and Charles Plott, "The Probability of a Cyclical Majority," Econometric, 38 (1970), pp.345, 354; and in Mark B. Garman and Morton I. Kamien, "The Pardox of Voting: Probability Calculations," Behavioral Science, 13 (1968), pp.306, 317; in William H. Riker, "Voting and the Summation of Preferences: An Interpretive Bibliographic Review of Selected Developments during the Last Decade," American

Political Science Review, 55 (1961); pp.900, 911. The development of these calculations are presented in David Klahr, "A Computer Simulation of the Paradox of Voting," American Political Science Review, 60 (1966); pp.384, 390.

4. The table is extracted from Amartya K. Sen's Collective Choice and Social Welfare, San Francisco, 1970; p.164.
5. See Amartya K. Sen's Collective Choice and Social Welfare, San Francisco, 1970; p.165.
6. The table is extracted from Amartya K. Sen's Collective Choice and Social Welfare, San Francisco, 1970; p.164.
7. See Amartya K. Sen's article, "A Possibility Theorem on Majority Decisions," Econometrica, 34 (1966); pp.491, 499.
8. See Alfred F. Mackay, Arrow's Theorem: The Paradox of Social Choice, 1980; pp.112, 113.
9. ibid., p.115.
10. See David H. Sanford's "Infinity and Vagueness", The Philosophical Review, 84 (1975); pp.534,35.
11. See Alfred F. Mackay, Arrow's Theorem: The Paradox of Social Choice, 1980; p.117.
12. ibid., pp.120, 121.
13. ibid., pp.123.

NOTES FOR CHAPTER 4

1. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.26.
2. ibid., p.27.
3. ibid., p.9.
4. ibid., p.28.
5. ibid., p.30.
6. Strictly speaking, the ranking computation is a purely formal operation on ordinal comparison. Therefore, rank-order method should not be interpreted as a 'cardinal' utility scaling.

NOTES FOR CHAPTER 5

1. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.96.
2. See J. M., and G. Tullock, The Calculus of Consent, University of Michigan Press, Ann Arbor, 1962; p.96.
3. For detail discussion of unanimity rule, see A. K. Sen's Collective Choice and Social Welfare San Francisco, 1970 chapter 2; pp.21,32.
4. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.81.
5. ibid., p.83.
6. Externalities can be positive. Here only concerns the negative externalities.
7. See A. K. Sen, Collective Choice and Social Welfare San Francisco, 1970; p.25.
8. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; pp.119,120.
9. One assumption I make here is that anyone who does not agree the rules of the game may choose not to play. In fact, even they participate in the games, they are playing different games, not the same game. This assumption, however, are unrealistic. But presumably if we are really at the earliest stage of the creation of a society, we must allow the choice of not participating, otherwise, individuals are forced to accept certain set of rules. Once the rules have been adopted, we may expected some policies will face opposition, but if these policies come out as a social choice through a set of 'unamity rules' , we than say that these policies are legitimate in the sense that they are produced by a set of rules of a system to which everyone consented.

NOTES FOR CHAPTER 6

1. See John Rawls, A Theory of Justice Cambridge, Mass.: Harvard University Press, 1971; pp.152,157.
2. See Stephen Strasnick, "The Problem of Social Choice: Arrow to Rawls." Philosophy and Public Affairs 5(3):pp.241, 30. Strasnick argues that Arrow's independence condition made the social choice unable to differentiate between situations in which individual preferences were the same but their priorities were different. Preference priority is morally relevant to social choice, therefore we have to take it into consideration.

NOTES FOR CHAPTER 7

1. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.19.
2. See William H. Riker and Peter C. Ordeshook, An introduction to Positive Political Theory, Prentice-Hall, 1973; pp.13, 14.
3. See Rogers Brubaker, The Limits of Rationality: An Essay on the Social and Moral Thought of Max Weber, George Allen & Unwin, London, 1984; ch. 3 and ch.4. From these chapters, a detail discussions about ethical rationality and irrationality can be found. A moral argument at least can be criticized as inconsistencies; this, can be seen as the necessary requirement for a man to be rational.
4. See Amo Tversky's article, "Intransitivity of Preferences," Psychological Review 76 [1969]:31-48. He points out that most often, we are confronted with complex multidimensional alternatives, it is difficult for an individual to choose 'rationally' He suggested that we might adopt "simplification procedures...which approximate one's 'true

preference' very well," but in some occasions, it might lead to intransitive preferences. This situation, is not the same as that individual is indifferent among alternatives.

5. This technique is called proof by contradiction. Suppose we want to prove p implies q. A direct proof would start by assuming p and then deriving q; but this may be very difficult. Since p implies q can fail only when p is true and q is false, we may try to show that this combination of p true and q is false can not happen. To do this, we start by assuming both that p is true and q is false and then deriving something that is clearly false, i.e. some contradictory results.

6. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.14.

7. See C. Dyke, Philosophy of Economics, Preceton Hill Press, 1981; p.15.

8. The example is from Larry S. Temkin's article, "Intransitivity and the Mere Addition Paradox", Philosophy and Public Affairs, 16, [1987], p.187. His calculation for the probability seems erroneous (I have consulted it with a graduate student in mathematics department). I present my calculation of probability for the three events here:

(1) For die A's six faces are 6, 6, 6, 2, 2, 2, and die B's are 5, 5, 4, 3, 3, 2:

Since $P(W_{AB}/_{B=2})=1/2$ and $P(B=2)=1/6$;

$P(W_{AB}/_{B=3})=1/2$ and $P(B=3)=2/6$;

$P(W_{AB}/_{B=4})=1/2$ and $P(B=4)=1/6$;

$P(W_{AB}/_{B=5})=1/2$ and $P(B=5)=2/6$;

(Where $P(W_{AB}/_{B=2})$ means the probability of A beating B when B occurs 2, and $P(B=2)$ means the probability of B occurs 2)

On any given roll, the probability of die A beating die B:

$P(W_{AB}) = 1/2x1/6 + 1/2x2/6 + 1/2x1/6 + 1/2x2/6$

$= 1/2[1/6 + 1/3 + 1/6 + 1/3]$

$= 1/2$ or 0.50

For $P(W_{BA}/_{A=2})=5/6$ and $P(A=2)=3/6$;

$P(W_{BA}/_{A=6})=0$ and $P(A=6)=3/6$;

Therefore, the probability of die B beating die A:

$P(W_{BA}) = 5/6x3/6 + 0x3/6$

$= 5/12$ or 0.416.

(2) For die B's six faces are 5, 5, 4, 3, 3, 2, and die C's are 6, 4, 3, 3, 3, 3:

Since $P(W_{BC}/_{C=3})=3/6$ and $P(C=3)=4/6$;

$P(W_{BC}/_{C=4})=2/6$ and $P(C=3)=2/6$;

$P(W_{BC}/_{C=6})=0$ and $P(C=6)=1/6$;

On any given roll, the probability of die B beating die C:

$P(W_{BC}) = 3/6x4/6 + 2/6x2/6 + 0x1/6$

$= 1/6[2 + 4/6]$

$= 14/36$ or 0.389

For $P(W_{CB}/_{B=2})=1$ and $P(B=2)=1/6$;

$P(W_{CB}/_{B=3})=2/6$ and $P(B=3)=2/6$;

$P(W_{CB}/_{B=4})=1/6$ and $P(B=4)=1/6$;

$P(W_{CB}/_{B=5})=1/6$ and $P(B=5)=2/6$;

Therefore, the probability of die C beating die B:

$P(W_{CB}) = 1x1/6 + 2/6x2/6 + 1/6x1/6 + 1/6+2/6$

$= 1/6[1 + 4/6 + 1/6 + 2/6]$

$= 1/6[13/6]$

$= 13/36$ or 0.36.

(3) For die C's six faces are 6, 4, 3, 3, 3, 3, and die C's are 6, 6, 6, 2, 2, 2:

Since $P(W_{CA}/_{A=2})=1$ and $P(A=2)=3/6$;

$P(W_{CA}/_{A=6})=0$ and $P(A=6)=3/6$;

On any given roll, the probability of die C beating die A:

$P(W_{CA})$ = 1x3/6 + 0x3/6

= 1/2

= 6/12 or 0.50

For $P(W_{AC}/_{C=3})$=3/6 and P(C=3)=4/6;

$P(W_{AC}/_{C=4})$=3/6 and P(C=4)=1/6;

$P(W_{AC}/_{C=6})$=0 and P(C=4)=1/6;

Therefore, the probability of die A beating die C:

$P(W_{AC})$ = 3/6x4/6 + 3/6x1/6 0x1/6

= 5/12 or 0.42.

9. See Amos Tversky's article on the topic, "Intransitivity of Preferences," Psychological Review, 76 [1969]: pp.31,48. Tversky observed that inconsistencies among repeated choices, he said, "[i]t seems, therefore, that the observed inconsistencies reflect inherent variability or momentary fluctuation in the evaluative process. This consideration suggests that preference should be defined in a probabilistic fashion. ... The inconsistency of the choices is thus incorporated into the preference relation as x is said to be preferred to y only when it is chosen over y more than half the time. Restating the transitivity axiom in terms of this definition yields.

P(x, y) $\geqslant$ ½ and P(x, y) $\geqslant$ ½ imply P(x, y) $\geqslant$ ½."; p.31

10. See Webster's Dictionary of the English Language.

11. See K. J. Arrow, Social Choice and Individual Values, John Wiley, New York, 1963; p.19.

12. See Peter Fishburn, "The Irrationality of Transitivity in Social Choice." Behavioral Science, (1970) 15(2): pp.119,23.

13. See Robert A. Dahl, A Preface To Democratic Theory, University of Chicago, 1956; p.37.

14. Arrow's first condition--unlimited domain,does not appear to be a rationality requirement. It does not seem that a limitation on individual or group preference orders is 'irrational' in any meaningful sense implied by this term. This condition represents a positive value in a liberal democratic culture we believe in freedom of choice. In

political philosophy, this might have been called the freedom of choice. Pareto principle and the condition of independence of irrelevant alternatives do not appear to be a rationality condition but rather an explication of the condition of popular sovereignty. Nondictarship is not a rationality condition but is the condition of political equality unheld in a populistic democracy.

15. See A.K. Sen, Collective Choice and Social Welfare, San Francisco, 1970; p.48.

16. See Robert A. Dahl, A Preface To Democratic Theory, University of Chicago, 1956; p.42.

17. See Alice Sturgis, Sturgis Standard Code of Parlimentary procedure, McGraw-hill, New York, 1966. p.136.

18. See Shackleton, Shackleton on The Law and Practice of Meetings, Sweet & Maxwell, London, 1977; p.49.

19. Paradox producing profile can be treated as a situation of tie vote. It can be seen more clearly in a cardinal scale voting method. See the last example in the section: A Preference Revealing Process.

NOTES FOR CHAPTER 8

1. The definition for Arrow's conditions can be found in his book, Social Choice and Individual Values, John Wiley, New York, 1963; pp. 13, 24, 26-30, 96, 97.

2. Sen called this consistency condition as Property α in his book, Collective Choice and Social Welfare, San Francisco, 1970; pp.16, 17. Sen says, "[t]o guarantee that properties of the choice function may have to be specified. ... Property α states that if some element of subset S_1 of S_2 is best in S_2, then it is best in S_1. This is a very basic requirement of rational choice". (Sen, p.17.) Kelly also proved that for total choice functions, transitive rationality implies path independence (his

theorem 3-6), and path independence implies Property α (his theorem 3-8). therefore, transitivity rationality implies Propertyα. It means that the Property α is the necessary condition for transitivity. The violation of Property α is the violation of transitivity. The Property α, can be viewed as a weaker requirement than transitivity for consistency. See Jerry S. Kelly, Arrow Impossibility Theorems, Academic Press, New York, Sen Fransico, 1978; pp.24, 26.

3. See Arrow's Social Choice and Individual Values, John Wiley, New York, 1963; p.46.
4. For the proof of Theorem 1, see Arrow's Social Choice and Individual Values, John Wiley, New York, 1963; pp.46, 48.
5. See Arrow's Social Choice and Individual Values, John Wiley, New York, 1963; p.48.

NOTES FOR CHAPTER 9

1. See K. J. Arrow, Social Choice and Individual Values, John Willey, New York, 1963; p.1.
2. See Douglas W. Rae, The Political Consequences of Electoral laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.17.
3. ibid., p.17.
4. The table is extracted from Arend Lijphart's book, Democracies: Patterns of Majoritarian and Consensus Government in Twenty-One Countries, Yale University Press, 1984; p.152. Data can also be obtained from Dick Leonard and Richard Natkiel's book, The Economist World Atlas of Elections: Voting Patterns in 39 Democracies, Hodder and Stoughton, 1987; p.9.
5. See Enid Lakeman, How Democracies Vote: A study of Electoral Systems, 4th edition, London, 1974; pp.63, 64.
6. Example is from Lakeman, p.86.

7. See Lakeman, p.110.
8. See Lakeman's Appendix II: "Examples of Elections under Proportional Systems"; pp.281, 286. And also Arend Lijphart's book, Democracies: Patterns of Majoritarian and Consensus Government in Twenty-One Countries, Yale University Press, 1984; pp.160, 165.
9. See Gudgin, G. and Taylor, P.J., "Electoral bias and the distribution of party voters", Transactions, Institute of British Geographers, 63 (1974), pp.53, 73.
10. See the section: "Accuracy of Representation" of the chapter VI: "The Single Transferable Vote" from Lakeman's book, How Democracies Vote: A study of Electoral Systems, 4th edition, London, 1974; pp.125, 128; and also see its Appendix II: "Examples of Elections under Proportional Systems"; pp.281, 286.

NOTES FOR CHAPTER 10

1. Besides Arrow's conditions, many scholars have proposed other social choice conditions for a social choice choice function. These conditions embody notions such as fairness, equity and consistency. Examples are: Anonymity, Neutrality, Monotonicity and Partition Consistency. See Sen, A.K., Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970; chapters 3, 5*, 9 and 9*.
2. See Brams, S. J. and P.C. Fishburn, "Deducing Preferences and Choices in the 1980 Presidential Election", Electoral Studies, 1 (1982); PP.333, 346. And also see P.C. Fishburn, "Social Choice and Pluralitylike Electoral Systems". In Arend Lijphart and Bernard Grofman (eds.), Electoral Laws and Their Political Consequences, Agathon Press Inc., New York; pp.193, 202.
3. See Dick Leonard and Richard Natkiel, The Economist World Atlas of Elections: Voting Patterns in 39 Democracies,

Hodder and Stoughton, 1987; p.2.

4. See Douglas W. Rae, The Political Consequences of Electoral Laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.24.

5. See Jack F. H. Wright, "Australian Experience with Majority-Preferential and Quota-Preferential Systems". In Arend Lijphart and Bernard Grofman (eds.), Electoral Laws and Their Political Consequences, Agathon Press Inc., New York; pp.126, 127.

6. Jack F. H. Wright points out that "[i]n earlier elections, there have been cases where majorities of seats have been won by parties with only minority support from the voters. In the House of Representatives election of 1954, the Labour party received over 50% of first preferences, but the Liberal-Country party coalition won a majority preferred to the coalition by more than half the voters, but on each occasion it failed to win half the seats in the House. There have also been elections of state houses which have given results of this kind. An example is the election of the House of Assembly of South Australia in 1968, when Labour candidates received 51.98% of the first preferences but the party won only 19 of the 39 seats." ibid. pp.128, 129.

7. See Arend Liphart, Rajael Lopez Pintor, and Yasunori Sone, "The Limited Vote and the Single Nontransferable Vote: Lessons from the Japanese and Spanish Examples". In Arend Lijphart and Bernard Grofman (eds.), Electoral Laws and Their Political Consequences, Agathon Press Inc., New York; pp.154, 169.

8. See Douglas W. Rae, The Political Consequences of Electoral Laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.28.

9. Example is from John Bonner's book, Introduction to the Theory of Social Choice, The John Hopkins University Press, 1986; p.91.

10. See Douglas W. Rae, The Political Consequences of Electoral

Laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.34.

11. Example is from Dick Leonard and Richard Natkiel's book, The Economist World Atlas of Elections: Voting Patterns in 39 Democracies, Hodder and Stoughton, 1987; p.2.
12. ibid., p.3.
13. See Douglas W. Rae, The Political Consequences of Electoral Laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.33.
14. George H. Hallet is one of the strong STVS advocates, he presents his reasons in his article, "Proportional Representation with the Single Transferable Vote: A Basic Requirement for Legislative Elections". In Lijphart and Grofman (eds.), Choosing an Electoral System: Issues and Alternatives, 1984; pp.113, 125.
15. See Douglas W. Rae, The Political Consequences of Electoral Laws, revised edition. New Haven, Conn.: Yale University Press, 1971; p.36.
16. ibid., p.37.
17. See Fishburn, P. C., and S. J. Brams, "Paradoxes of preferential voting", Mathematics Magizine 56 (1983) p.207. Fishburn identifies this case as "Multiple-Districts Paradox". It violates the partition consistency proposed by Young. (P. H. Young, "An Axiomatization of the Borda Rule", Journal of Economic Theory, 9 (1974), pp.43, 52.) The desirability of this consistency requirement for democracy can be reasoned as follows: one would presumably expect that a candidate winning in two separate districts would also be the winner when a combined score of the two districts was calculated, given that the preferences of the voters in the two districts remained identical in the enlarged one. Clearly if one receives most of the votes in two separate groups, he or she should also receive most of the votes from the combined group. In this sense, a electoral system that fails to meet this condition may be considered as inconsistent.

NOTES FOR APPENDIX

1. See White Paper: The Development of Representative Government: The Way Forward, February 1988, Printed by the Government Printer, Hong Kong; pp.20, 21.
2. ibid., Appendix C, Example B; pp.26, 28.
3. Exhausted votes are those votes fail to record the next preference. Thus, votes are not transferable.

BIBLIOGRAPHY AND REFERENCES:

Arrow, K. J. (1963): Social Choice and Individual Values. 2nd edition. New York: Wiley.

-----(1967): "Values and Collective Decision Making", Philosophy, Politics and Society, 3 (Peter Laslett and W. G. Runciman, eds.) pp. 215, 232. Oxford: Blackwell, 1967.

-----(1977): Social Choice and Justice: Collected Papers of Kenneth J. Arrow, Basil Blaskwell Publisher.

Black, D. (1958): The Theory of Committees and Elections, Cambridge: Cambridge University Press.

-----(1969): "On Arrow's Impossibility Theorem", Journal of law and Economics, 12, October, 1969; pp. 227, 248.

Black, J. H. (1978): "The Multicandidate Calcus of Voting: Application to Canadian Federal Elections", American Journal of Political Science, 22, pp. 609, 638.

-----(1980): "The Probability-Choice Perspective in Voter Decision Making Models", Public Choice, 35 (5), pp.565, 574.

Blair, D. H. (1975): "Path-Independent Social Choice Functions: A Further Result", Econometrica, 43, January, 1975; pp.173, 174.

Blair, D. H., G. Bordes, J. S. Kelly, and K. Suzumura (1976): "Impossibility Theorems without Collective Rationality", Journal of Economic Theory, 13, December, 1976; pp.361, 379.

Blau, J. H. (1957): "The Existence of a Social Welfare Function", Econometrica, 25, 1957; pp.302, 313.

-----(1971): "Arrow's Theorem with Weak Independence", Economica, 38, 1971; pp. 413, 420.

-----(1972): "A Direct Proof of Arrow's Theorem", Econometrica, 40, 1972; pp. 61, 67.

Bogdanor, V. and D. Butler (eds.) (1983): Democracy and Elections: Electoral Systems and Their Political Consequences. Cambridge: Cambridge University Press.

Bonner, J. (1986): Introduction to the Theory of Social Choice, The John Hopkins University Press.

Brams, S. J. and P. C. Fishburn (1982): "Deducing Preferences and Choices in the 1980 Presidential Election", Electoral Studies, 1; pp.333, 346.

Brams, S. J. and P. C. Fishburn (1984a): "Some Logical Defects of the Single Transferable Vote". In Lijphart and Grofman (eds.), Choosing an Electoral System: Issues and Alternatives; pp.238, 250.

Brams, S. J. and P. C. Fishburn (1984b): "A variable-size Legislature to Achieve Proportional Representation. In Lijphart and Grofman (eds.), Choosing an Electoral System: Issues and Alternatives; pp.270, 275.

Brown, D, J., (1975): "Aggregation of Preferences", Quarterly Journal of Economics, 89, August, 1975; pp.456, 469.

Brubaker, R. (1984): The Limits of Rationality: An Essay on the Social and Moral Thought of Max Weber, George Allen & Unwin, London.

Bruce, B. (1972): "Toward an Estimate of the Frequency of Occurance of the Paradox of Voting in U.S. Senate Roll Call

Votes". In R. Niemi and H. Weisberg, (eds.), Probability Models of Collective Decision-making, Columbus, Ohio: Merrill, 1972.

Buchanan, J. M. and G. Tullock (1962): The Calculus of Consent. Ann Arbor: The University of Michigan Press.

Campell, C. D., and G. Tullock. (1966): "The Paradox of Voting--a Possible Method of Calculation", American Political Science Review, 60, September, 1966, pp.684, 685.

Campell, D. E. (1973): "Social Choice and Intensity of Preference", Journal of Political Economy, 81, 1973; pp.211, 218.

-----(1976): "Democratic Preference Functions", Journal of Economic Theory, 12, 1976; pp.259, 272.

Chernoff, H. (1954): "Rational Selection of Decision Functions", Econometrica, 22, October, 1954; PP.422, 443.

Clarke, E. H. (1971): "Multipart pricing of public goods", Public Choice, 11, 1971; pp.17, 33.

Coleman, J. S. (1966): "The Possibility of a Social Welfare Function", American Economic Review, 56, December, 1966; pp.1105, 1122.

Contini, B. (1966): "A Note on Arrow's Postulates for a Social Welfare Function", Journal of Political Economy, 74, June, 1966; pp.278, 280.

Dahl, R. A. (1956): A Prefare To Democratic Theory, University of Chicago.

Davis, J. M. (1958): "The Transitivity of Preferences". Behavioral Science, 3, January, 1958, pp.26, 33.

DeMeyer, F. and Plot, C. (1970): "The Probability of a Cyclical Majority", Econometrica, 38, March, 1970; pp.345, 354.

Diamond, P. A. (1967): "Cardinal Welfare, Individualistic Ethics and Interpersonal Comparisons of Utility: Comment", Journal of Political Economy, 75, October, 1967; pp.765, 766.

Dubins, L. (1977): "Group Decision DeVices", American Mathematical Monthly 84, 1977; pp.350, 356.

Dyke, C. (1981): Philosophy of Economics, Preceton Hill Press.

Fishburn, P. C. (1969): "Preferences, Summation and Social Welfare Functions" Management Science, 16, November, 1969; pp.119, 123.

-----(1970): "Arrow's Impossibility Theorem: Concise Proof and Infinite Voters", Journal of Economic Theory, 2, March, 1970; pp.103, 106.

-----(1970): The Irrationality of Transitivity in Social Choice", Behavioral Science, 15, March, 1970; pp.103, 106.

-----(1970): "Intransitive Individual Indifference and Transitive Majorities", Econometrica, 38, May, 1970; pp.482, 489.

-----(1971): "Should Social Choice Be Based on Binary Comparisons?" Journal of Mathematical Sociology, 1, January, 1971, pp.133, 142.

-----(1973): The theory of Social Choice, Princeton, New Jersey: Princeton University Press.

-----(1973): "Summation Social Choice Functions", Econometrica, 41, November, 1973; pp.1183, 1196.

-----(1974): "On Collective Rationality and a Generalized

Impossibility Theorem", Review of Economic Studies, 41, October, 1974; pp.445, 459.

-----(1983): "Dimensions of Election Procedures: Analyses and Comparisons", Theory and Decision, 15; pp.371, 397.

-----(1986): "Social Choice and Pluralitylike Electoral systems". In Arend Lijphart and Bernard Grofman (eds.), Electoral Laws and Their Political Consequences, Agathon Press Inc., New York; pp.193, 202.

Fishburn, P. C., and S. J. Brams (1982): "Deducing Preferences and Choices in the 1980 Presidential election", Electoral Studies, 1, 1982; pp.333, 346.

Fleming, J. M. (1952): "A Cardinal Concept of Welfare", Quarterly Journal of Economics, 66, August, 1952; pp.366, 384.

Groves. T. and J. Ledyard. (1977): "Optimal Allocation of Public Goods: a soulution to the 'free rider roblem'", Econometrica , 45. 1977; pp.783, 810.

Gudign, G. and P. J. Taylor. (1974): "Electoral Bias and the Distribution of party Voters', Transactions, Institute of British Geographers, 63, 1974; pp.53, 73.

Hammond, P. J. (1976): "Equity, Arrow's Conditions and Rawls' Difference Principle", Econometrica, 44, July, 1976; pp.793, 804.

Harsanyi, J. C. (1955): "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility", Journal of Political Economy, 63, August, 1955; pp.309, 321.

Kelly, J. S. (1974): "Necessity Conditions in Voting Theory", Journal of Economic Theory, 8, June, 1974; pp.149, 160.

-----(1976): "Algebraic Results on Collective Choice Rules", Journal of Mathematical Economics, 3, December, 1976; pp.285, 293.

-----(1978): Arrow Impossibility Theorems, New York: Academic Press.

Hong Kong Government (1988): The Development of Representative Government: The Way Forward, February 1988, Government Printer, Hong Kong.

Lakeman, E. (1974): How Democracies Vote: A Study of Majority and Proportional Electoral Systems, London: Faber and Faber.

Leonard, D., and Natkiel R. (1987): The Economist World Atlas of Elections: Voting Patterns in 39 Democracies, Hodder and Stoughton.

Leuthold, D. A. (1968): Electioneering in a Democracy, New York: Wiley.

Lijphart, A. (1984): "Advances in the Comparative Study of Electoral Systems", World Politics, 36; pp.424, 436.

Little, I. M. D. (1952): "Social Choice and Individual Values", Journal of Political Economy, 60, October, 1952; pp.422, 432.

Mackay, A. F. (1973): "A Simplified Proof of an Impossibility Theorem", Philosophy of Science, 40, June, 1973; pp.175, 177.

-----(1980): Arrow's Theorem: The Paradox of Social Choice", Yale University Press.

May, K. O. (1954): "Intransitivity, Utility and the Aggregation of Preference Patterns", Econometrica, 22, January, 1954; pp.1, 13.

May, R. M. (1971): "Some Mathematical Remarks on the Paradox of Voting", Behavioral Science, 16, March, 1971; pp.143, 151.

Neumann, J. V. and O. Morgenstern. (1953): The Theory of Games and Economic Behavior, Princeton, New Tersey, Princeton University Press, 1953.

Niemi, R. G. and Weisberg H. (1968) "A Mathematical Solution for the Probability of the Paradox of Voting", Behavioral Sciences, 13, July, 1968; pp.317, 323.

-----(1976): "The Choice of Voting Systems", Scientific American, 234, June, 1976; pp.21, 27.

Osborne, D. K. (1976): "Irrelevant Alternatives and Social Welfare", Econometrica, 44, September, 1976; pp.1001, 1015.

Plott, C. R. (1967): "A Notion of Equilibrium and its Possibility under Majority Rule", American Economic Review, 57, 1967; pp.787, 806.

-----(1973): "Path Independence, Rationality and Social Choice", Econometrica, 41; pp.1075, 1091.

-----(1976): "Axiomatic Social Choice Theory: An Overview and Interpretation", American Journal of Political Science, 20, August, 1976; pp.511, 596.

Rae, D. W. (1975): "Maximin Justice and an Alternative Principle of General Advantage", American Political Science Review, 69; pp.630, 647.

-----(1971): "The Political Consequences of Electoral Laws", New Haven: Yale University Press.

-----(1971): "Thresholds of Representation and Thresholds of Exclusion", Comparative Political Studies, 3; pp.479, 488.

Rawls, J. (1971): A Theory of Justice, Cambridge, Mass: Harvard University Press, 1971; pp.152, 157.

Ray, P. (1973): "Independence of Irrelevant Alternatives", Econometrica, 41; pp.987, 991.

Richelson, J. T. (1975): "A Comparative Analysis of Social Choice Functions", Behavioral Science, 20; pp.331, 337.

-----(1978): "A Comparative Analysis of Social Choice Functions, II", Behavioral Science, 23; pp.38, 44.(a)

-----(1978): "A Comparative Analysis of Social Choice Functions, III", Behavioral Science, 23; pp.169, 176.(b)

Riker, W. (1961): "Voting and the Summation of Preferences", American Political Science Review, 55; pp.900, 911.

-----(1965): "Arrow's Theorem and Some Examples of the Paradox of Voting", in Mathematical Applications in Political Science, (John, M. C. ed.), pp.41, 60. Dallas, Texas: Southern Methodist University Press.

Riker, W. H., and P. C. Ordeshook. (1968): "A Theory of the Calculus of Voting", American Political Science Review, 62; pp.25, 42.

Rokkan, S. (1968): "Elections: Electoral Systems". In D. L. Sills (ed.), International Encyclopedia of the Social Sciences, Vol. 5, pp.6, 21. New York: Macmillan and Free Press.

Rose, R. (1983): "Elections and Electoral Systems: Choices and Alternatives". In Bogdanor and Butler (eds.), Democracy and

Elections: Electoral Systems and Their Political Consequences. Cambridge: Cambridge University Press; pp.20, 45.

Rosensweig, J. A. (1981): "Highest Average Methods of Allocating Seats Under Proportional Representation: A Clarification", Political Studies, 29; pp.279, 281.

Salmon, W. C. (1966): The Foundations of Scientific Inference, University of Pittsburgh Press, 1966.

Sanford, D. H. (1975): "Infinity and Vagueness", The Philosophical Review, 84, 1975; pp.534, 35.

Satterthwaite, M. A. (1975): "Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", Journal of Economic Theory, 10; pp.187, 217.

Schwartz, T. (1970): "On the Possibility of Rational Policy Evaluation", Theory and Decision, 1, 1970; pp.89, 106.

-----(1972): "Rationality and the Myth of the Maximum", Nous, 6; pp.97, 117.

Sen. A. K. (1966): "A Possibility Theorem on Majority Decisions", Econometrica, 34, 1966; pp.481, 499.

-----(1969): "Quasi-transitivity, Rational Choice and Collective Decisions", Review of Economic Studies, 36; pp.381, 393.

-----(1970): Collective Choice and Social Welfare, San Francisco: Holden-Day.

-----(1970): "Interpersonal Comparison and Partial Comparability", Econometrica, 38, May, 1970; pp.393, 409.

-----(1972): "Interpersonal Comparison and Partial Comparability: A Correction", Econometrica, 40, September, 1972; p.959.

-----(1976): "Liberty, Unanimity and Rights", Economica, 43, August, 1976; pp.217, 245.

-----(1977): "Social Choice Theory: A Re-Examination", Econometrica, 45; pp.53, 89.

Shackleton (1977): Shackleton on The Law and Practice of Meetings, Sweet & Maxwell, London.

Smith, J. H. (1973): "Aggregation of Preferences with Variable Electorate", Econometrica, 41; pp.1027, 1041.

Strasnick, S. (1976): "Social Choice and the Derivation of Rawls' Difference Principle", The Journal of Philosophy, 73, February, 1976; pp.85, 99.

-----(1976): "The Problem of Social Choice: Arrow to Rawls", Philosophy and Public Affairs, 5, 1976; pp.241, 273.

Stratmann, W. C. (1974): "The Calculus of Rational Choice", Public Choice, 18; pp.93, 105.

Sturgis, A. (1966): Sturgis Standard Code of Parlimentary Procedure, McGraw-hill, New York.

Taylor, P. J., and R. J. Johnston (1979): Geography of Elections, London: Penguin.

Tideman, T. N. and G. Tullock. "A New and Superior Process for Making Social Choices", Journal of Political Economy, 84, 1976; pp.1145, 1159.

Tullock, G. and J. Buchanan. (1962): The Calculus of Consent,

University of Michigan Press, Ann Arbor, 1962.
-----(1967): "The General Irrelevance of the General Possibility Theorem", Quarterly Journal of Economics, 81, May, 1967; pp.256, 270.
Tversky, A. (1969): "Intransitivity of Preferences", Psychological Review, 76, 1969; pp.31, 48.
Vickrey, W. (1960): "Utility, Strategy and Social Decision Rules", Quarterly Journal of Economics, 74, November, 1960; pp.507, 535.
Waldner, I. (1972): "The Empirical Meaningfulness of Interpersonal Utility Comparisons", Journal of Philosophy, 69, February, 1972; pp.87, 103.
-----(1973): "The Possibility of Rational Policy Evaluation", Theory and Decision, 4, September, 1973; pp.85, 90.
Weaver, L. (1984): "Semi-Proportional and Proportional Representation Systems in the Untied States". In Lijphart and Grofman (eds.), Choosing an Electoral System: Issues and Alternatives, pp.191, 206.
Wesley C. S. (1966): The Foundations of Scientific Inference, University of Pittsburgh Press, 1966; pp.84, 87.
Williamson, O. E., and Thomas J. S. (1967): "Social Choice: A Probabilisitc Approach", Economic Journal, 77, December, 1967; pp.797, 813.
Wilson, R. (1972): "Social Choice Theory without the Pareto Principle", Journal of Economic Theory, 5, December, 1972; pp.478, 486.
-----(1975): "On the Theory of Aggregation", Journal of Economic Theory, 10, February, 1975; pp.89, 99.
Young, P. H. (1974): "An Axiomatization of the Borda Rule", Journal of Economic Theory, 9, September, 1974; pp.43, 52.
-----(1974): "A Note on Preference Aggregation", Econometrica, 42, November, 1974; pp.1129, 1131.